# MAT2041 Tutorial

## Week 10

The Chinese University of Hong Kong, Shenzhen

Nov. $21^{st}, 2023 \sim$ Nov. $24^{th}, 2023$

# Outline

Definition Suppose $a_1, a_2, \ldots, a_n$ are given real numbers, $\mathrm{x} = (x_1, x_2, \ldots, x_n)$, then

$$f(\mathrm{x}) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

is called a **linear function** from $\mathbb{R}^n$ to $\mathbb{R}$.

Alternative Suppose $\mathrm{a} \in \mathbb{R}^n$ is a given real vector, $\mathrm{x} \in \mathbf{R}^n$, then

$$f(\mathrm{x}) = \langle \mathrm{x}, \mathrm{a} \rangle$$

is called a **linear function** from $\mathbb{R}^n$ to $\mathbb{R}$.

# Exercise 1

Suppose $V = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a subspace of $\mathbb{R}^3$.

(1) Find a non-zero linear function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ that: $\forall x \in V, \varphi(x) = 0$

(2) Prove that $\text{Ann}(V) = \{\varphi \mid \forall v \in V, \varphi(v) = 0\}$ is a linear space.

# Review

## Linear Transformation

**Definition** Suppose $a_{ij}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are given real numbers, $\mathrm{x} = (x_1, x_2, \ldots, x_n)$, then

$$f(\mathrm{x}) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}$$

is called a **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Alternative 1** Suppose $A \in \mathbb{R}^{m\times n}$ is a given real matrix, $\mathrm{x} \in \mathbf{R}^n$, then

$$f(\mathrm{x}) = A\mathrm{x}$$

is called a **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Alternative 2** If a mapping $f$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfies

$$f(\alpha\mathrm{x} + \beta\mathrm{y}) = \alpha f(\mathrm{x}) + \beta f(\mathrm{y}), \forall\alpha, \beta \in \mathbb{R}, \forall\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$$

Then $f$ is a **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$.

# Exercise 2

Consider linear transformation $L : V \to W$, where $V, W$ are linear spaces.
Define $\text{Null}(L) \triangleq \{y \in V : L(y) = 0\}$.

Prove that $\text{Null}(L) = \{0\}$ **if and only if** the following holds:
$\forall v_1, v_2 \in V$, $v_1 \neq v_2$ can imply that $L(v_1) \neq L(v_2)$.

# Exercise 3

Till now, we are talking about linear transformations over vectors. In fact, the same definition can be applied to matrices, polynomials, etc.

(1) Let $T_1 : \mathbb{P}_2 \to \mathbb{P}_2$ be the transformation $T_1(p) = p'(x) - p(x)$, where $\mathbb{P}_2$ is a subspace for polynomials of the order not greater than 2. Prove that $T_1$ is a linear transformation.

(2) Let $T_2 : \mathbb{P}_2 \to \mathbb{R}$ be the transformation $T_2(p) = p'(5) - p(3)$. Prove that $T_2$ is a linear transformation.

# Exercise 4

Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, find a matrix $A$ such that $T(v) = Av$ for all $v \in \mathbb{R}^2$.

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

.

# Solution

## Exercise 4

Consider one way to solve *Exercise 4*( $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \quad (x, y \in \mathbb{R})$ ):

$T\begin{pmatrix} x \\ y \end{pmatrix}$ contains two independent parts: $x$ and $y$. And

$$T\begin{pmatrix} x \\ y \end{pmatrix} = x \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Why? Each column of $A$ is just representing the “effect” of one variable. To be specific, it is the effect of each element in the **basis** of the original space.

# Step Further

## Matrix representation

Actually, we can write any $T\begin{pmatrix} x \\ y \end{pmatrix}$ by the linear combination of:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad ; \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$T(\alpha_1 c_1 + \ldots + \alpha_n c_n) = \alpha_1 T(c_1) + \ldots + \alpha_n T(c_n) = [T(c_1), \cdots T(c_n)] \cdot \alpha$$

**ONLY the basis matters!**

**Naive Strategy**:

1. Find a basis $\mathcal{B}$ of the original linear space.
2. $\forall i$, find the vector $T(\mathcal{B}_i)$ (the "effect")
3. Combine them together to form a matrix.

Note that then when calculating the linear transformation of $x$, $x$ should be represented by the basis (coordinate vector), so that $Ax$ makes sense.

## Step Further

### Matrix representation

Recall in *Exercise 3*, linear transformation can be define on spaces other than real vectors, like polynomials.

Easy We can still find a basis for the original space. And we can find the coordinate vector for any $x$.

Hard We cannot find the "effect" using a vector, then we cannot combine them to a matrix.

**Solution**: Find a basis for the target space as well! Then we can represent the transformed value in its coordinate vector, thus we can form a matrix.

$$\begin{cases} T\begin{pmatrix}1\\0\end{pmatrix} = -1e_1 + 0e_2 \\ T\begin{pmatrix}0\\1\end{pmatrix} = 0e_1 + 1e_2 \end{cases} \implies A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

# Step Further

## Matrix representation

**Definition** For linear transformation $T : V \to W$, where $V$ has a basis $\mathcal{A}$, $W$ has a basis $\mathcal{B}$. If matrix $\mathcal{C} = (c_{ij})$ satisfies $T(\mathcal{A}_j) = \sum_{j=1}^{m} c_{ij}\mathcal{B}_i$, then $\mathcal{C}$ is the matrix representation of $T$ under bases $\mathcal{A}, \mathcal{B}$.

**General Strategy: Find a matrix representation** $M$:

1. Find a basis $\mathcal{A}$ for $V$.
2. Find a basis $\mathcal{B}$ for $W$.
3. $\forall i$, Find $T(\mathcal{A}_i)$ as a linear combination of elements in $\mathcal{B}$.
4. Write it as a **column** vector (the coordinate vector of $T(\mathcal{A}_i)$).
5. Combine them to a matrix.

# Exercise 5

Recall HW4, Problem 6. Suppose linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is defined as $T(X) = AX - XA,\ X \in \mathbb{R}^{2\times 2}, A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

1. Find a basis $\mathcal{B}$ for $\mathbb{R}^{2\times 2}$.
2. For each $\mathcal{B}_i \in \mathcal{B}$, write $T(\mathcal{B}_i)$ as a linear combination of elements in $\mathcal{B}$.
3. Find the matrix representation of $T$ under the basis you found.

# Acknowledgment

These slides are based on previous tutorial materials of MAT2041. We thank previous TAs and instructors for sharing previous course materials.