MAT2041 Tutorial

Week 11

The Chinese University of Hong Kong, Shenzhen

Nov. $28^{th}, 2023 \sim$ Dec. $1^{st}, 2023$

Outline

Exercise 1

1. Consider a 120° rotation around the axis $x = y = z$. Show that the vector $u = (1, 0, 0)$ is rotated to the vector $v = (0, 1, 0)$ (similarly v is rotated to $w = (0, 0, 1)$ and w is rotated to u).
2. How is $v - u$ related to the vector $(1, 1, 1)$ along the axis?
3. Find the matrix A that produces the rotation (i.e. Av is the rotation of v), and explain why $A^3 = I$.
4. Find the (complex) eigenvalues of A.

Review

Eigenvalue, Eigenvector and Eigenspace

Definition Let $A \in \mathbb{C}^{n \times n}$ be a square matrix. If there exists a scalar $\lambda \in \mathbb{C}$ and a **nonzero** vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x$$

then λ is called an **eigenvalue** and x is called an **eigenvector** with respect to λ.

Then the set $\{x \mid (\lambda I - A)x = 0\}$ forms a linear space, called the **eigenspace** w.r.t. λ.
(Note that the eigenspace w.r.t. λ is (the set of all eigenvectors) $\cup\{0\}$)

Definition

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

is called the **characteristic polynomial** of A, and the equation $p_A(\lambda) = 0$ is called the **characteristic equation**.

Exercise 2

Find or compute the following expression when $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}$:

1. characteristic polynomial
2. characteristic equation
3. eigenvalues

Exercise 3

Show that the eigenvalues of a triangular matrix are exactly the diagonal entries.

Exercise 4

Let $A \in \mathbb{R}^{n \times n}, X \in \mathbb{R}^{n \times k}, B \in \mathbb{R}^{k \times k} (k \leq n)$.
If $AX = XB$, and X has full column rank, show that any eigenvalue of B is also an eigenvalue of A.

(The column space $C(X)$ is called an **invariant subspace of** A since $A(C(X)) \subseteq C(X)$, where A is considered as a linear transformation.)

Exercise 5

Let $A \in \mathbb{R}^{n \times n}, \lambda = \alpha + \mathrm{i}\beta (\beta \neq 0)$ is an eigenvalue of A. Show that $\bar{\lambda} = \alpha - \mathrm{i}\beta$ is also an eigenvalue of A.

Acknowledgment

These slides are based on previous tutorial materials of MAT2041. We thank previous TAs and instructors for sharing previous course materials.

This is the last tutorial of this semester. Thank you all for your active participation and sincere suggestions over the past three months.

Wish you good luck in your final exam.