MAT2041 Tutorial

Week 6

The Chinese University of Hong Kong, Shenzhen

Oct. $24^{th}, 2023 \sim$ Oct. $27^{th}, 2023$

Outline

Review

Block Matrix Inverse and Block LU decomposition

Above all, let's reemphasize that: In homeworks or exams, if you want to show matrix B is the inverse of A, you should verify both sides: $BA = AB = I$. (unless there are extra instructions)

LU decomposition Assume $E_k E_{k-1} \cdots E_1 A = U$, where U is an upper triangular matrix and $\forall i, E_i$ is an elementary matrix.
Then $A = (E_k E_{k-1} \cdots E_1)^{-1} U = LU$.

Inverse of Block Upper Triangular Matrix
Recall that the inverse of an upper triangular matrix U is

$$U^{-1} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

Block Elementary Row operation (Not mentioned in lectures before)

For block matrices, block elementary row operations can be defined. We have

1. switch two rows. Examples omitted.
2. **left**-multiply a row by a **matrix**. E.g. $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \to \begin{bmatrix} PA & PB \\ C & D \end{bmatrix}$
3. add **the product of a matrix and a row** to another row.
   E.g. $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \to \begin{bmatrix} A & B \\ PA + C & PB + D \end{bmatrix}$

Block Elementary Matrix (Not mentioned in lectures before)

Accordingly, block elementary matrices can be defined.

E.g. matrix $\begin{bmatrix} I_1 & 0 \\ P & I_2 \end{bmatrix}$ represent the operation of "add **matrix P left-multiply the first row** to **the second row**".

We can prove it since $\begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ PA + C & PB + D \end{bmatrix}$.

Exercise 1

Given 2×2 **invertible** block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is also invertible. Find M^{-1}.

Hint: Think about the content we reviewed just now:

1. Block elementary matrix
2. LU decompostion (Can you extend this to the case of block matrices?)
3. Inverse of block upper triangular matrix

(Rectangular) Matrix in RREF

$$\begin{bmatrix} 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix}$$

Column exchanges Exchanging columns of A can lead to the form $\begin{bmatrix} I_k & F \\ 0 & 0 \end{bmatrix}$

Solutions Express pivot variables by free variables.

E.g.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{cases} x_1 = 3 - x_3 \\ x_2 = 2 + x_3 \end{cases} \quad \to S = \left\{ \begin{bmatrix} 3-t \\ 2+t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

*Trick when an $m \times n$ matrix A is expressed as $\begin{bmatrix} I_k & F \\ 0 & 0 \end{bmatrix}$, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} -F \\ I_{n-k} \end{bmatrix} \cdot \vec{\lambda} \quad (\vec{\lambda} \in \mathbb{R}^n)$$

Exercise 2

Solve the linear system

$$\begin{cases} x_1 - 3x_2 + 2x_3 + x_4 = 6 \\ x_3 + 5x_4 = 3 \\ x_1 - 3x_2 + 3x_3 + 6x_4 = 9 \end{cases}$$

Informal interpretation Linear space is a set that

- is equipped with addition and scalar multiplication;
- any linear combination of elements is in this set.

Formal definition V is called a linear space over $\mathbb{R}$ if the 8 axioms hold:

(A1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \forall \mathbf{u}, \mathbf{v} \in V$
(A2) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+\mathbf{v}+\mathbf{w}, \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
(A3) There exists a element $\mathbf{0}$ s.t. $\mathbf{u}+\mathbf{0}=\mathbf{u}, \forall \mathbf{u} \in V$.
(A4) If $\mathbf{u} \in V$, then there exists $-\mathbf{u}=(-1)\mathbf{u}$, s.t. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
(A5) $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}, \forall \alpha \in \mathbb{R}, \mathbf{u}, \mathbf{v} \in V$.
(A6) $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}, \forall \alpha, \beta \in \mathbb{R}, \mathbf{u} \in V$.
(A7) $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}, \forall \alpha, \beta \in \mathbb{R}, \mathbf{u} \in V$.
(A8) $1\mathbf{u}=\mathbf{u}$

Proposition Suppose V is a linear space. W is a subspace of V if:

- W is a **subset** of V.
- $0 \in W$.
- $\forall u, v \in W : u + w \in W$.
- $\forall u \in W, \alpha \in \mathbb{R} : \alpha u \in W$.

Exercise 3

The trace $\operatorname{tr} : \mathbb{R}^{n \times n} \to \mathbb{R}$ of an $n \times n$ matrix is defined by summing the main diagonal:

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

The subset of trace-free matrices is denoted

$$\mathfrak{sl}_n(\mathbb{R}) = \left\{ A \in \mathbb{R}^{n \times n} : \operatorname{tr} A = 0 \right\}$$

Show that $\mathfrak{sl}_n(\mathbb{R})$ is a subspace of $\mathbb{R}^{n \times n}$

Definition - span Suppose is V a linear space, $\mathcal{U} = \{u_1, u_2, \ldots, u_k\}$ is a subset of V. Then
$\text{span}(\mathcal{U}) = \{\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}$

Definition - spanning set Suppose is V a linear space, $\mathcal{U} = \{u_1, u_2, \ldots, u_k\}$ is a subset of V. If $\text{span}(\mathcal{U}) = V$, then $\mathcal{U}$ is **a spanning set** of V, or $\mathcal{U}$ **spans** V

Excercise 4

Recall that we have shown in our lectures that P_2, the set of polynomials with degrees ≤ 2, is a linear space.
Show that $S = \left\{1 + x^2, 2 - x^2, x, 1 + 4x\right\}$ spans the linear space $P_2(\mathbb{R})$.

Null Space The solution set of a homogeneous linear system $Ax = 0$ is a linear space, denoted as $N(A)$; i.e. $N(A) = \{x \mid Ax = 0\}$

Column Space Suppose $A = \{\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n}\} \in \mathbb{R}^{m \times n}$ is a matrix.
Then span($\{\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n}\}$) is called the column space of A, denoted as $C(A)$.

Exercise 5

Show that

$$N(A^T A) = N(A)$$

Acknowledgment

These slides are based on previous tutorial materials of MAT2041. We thank previous TAs and instructors for sharing previous course materials.