Welcome to the tutorial session of week7！！！

IN THIS TUTORIAL SESSION, WE WILL FOCUS ON:

1. Solution set of Ax=b
2. Linear independence
3. Basis
4. Rank

1. Solve a General linear system Ax=b

(1)Doing Gaussian elimination
(2)Compute null space $X_{special}$ of Ax=0. Express pivot variables as forms of free variables
(3)Compute a particular solution X_p to Ax = b
(4)The solution to Ax=b is $X_{complete} = X_p + X_{special}$

Exercise1

1. For the following augmented matrix
(1) Compute the RREF form
(2) Write the solution to Ax=b

$$A|b = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 8 & 10 & 8 \\ 3 & 3 & 10 & 13 & 10 \end{array}\right]$$

2. Linear dependence

Claim 13.1 (linear dependence and span)
Suppose V is a linear space over $\mathbb{R}$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.
If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear dependent, then there exists $t \in \{1, \ldots, k\}$ such that:

i) $\mathbf{u}_t$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_k$.
ii) span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} =$ span$(\{\mathbf{u}_1, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_k\})$

Corollary: If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linear independent, then span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ can NOT be simplified (i.e. expressed as the span of $k - 1$ elements)

Exercise2

Let x_1, x_2,andx_3 be linearly independent vectors in R^n. Show that if x_4 is a vector in R^n that is not in Span$\{x_1, x_2, x_3\}$,thenx_1, x_2, x_3,and x_4 are linearly independent.

3. Basis and dimension

(1) Definition of basis

Suppose V is a linear space over R. Suppose $\alpha\{u_1, u_2, \cdots, u_k\} \subseteq V$. We say U is a basis if

(i) $u_1, u_2, \cdots, u_k$ are linearly independent.

(ii span$(U) = V$.

(2) Definition of dimension

Suppose V is a linear space. If V has a basis U with n elements, then we say the dimension of V is n, denoted as dim(V)$= n$, or V is n-dimensional.

4. Rank

(1) Definition of rank

The rank of matrix A is defined as the number of nonzero pivots in RREF form of A.

(2) Column rank and Row rank

Theorem 7.6 The row space and column space both have the same dimension r. We call $\dim(C(A))$ as column rank; $\dim(\mathcal{R}(A))$ as row rank. In brevity, column rank=row rank= rank, i.e.,

$$\dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A)) = \text{rank}(A)$$

Exercise3

For matrix $A \in R^{n \times n}$, we have Ax=b. Prove that this linear system has a solution iff r(A) = r([A,b])

Exercise4

$$r(A) + r(C) \leq r\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \leq r(A) + r(B) + r(C)$$

Exercise5

$A \in R^{p\times n}, B \in R^{n\times q}, \text{Prove that} \quad r(A) + r(B) \leqslant h + r(AB)$

Exercise6

Given s matrices $A_1, ..., A_s \in R^{n\times n}$, assuming $A_1 A_2 ... A_s = 0$. Show that $r(A_1) + r(A_2) + ... + r(A_s) \leq n(s-1)$

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