# Assignment 1: Vector Operations and Gaussian Elimination

**Academic Integrity Reminder:** Please ensure that you complete this assignment independently. If your solution appears to be identical or highly similar to others or generated using AI, you may be required to meet with the TA to verify your understanding. Cases of academic dishonesty may result in penalties, including a zero score for the assignment or other actions by the disciplinary committee. If you have any questions or need assistance, feel free to reach out to the course staff for guidance. We encourage you to engage in the learning process with integrity and responsibility

1. Verify that the dot product defined in the lecture satisfies the following properties (10pts):

a) Linearity: $\langle a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w}\rangle = a\langle \boldsymbol{u}, \boldsymbol{w}\rangle + b\langle \boldsymbol{v}, \boldsymbol{w}\rangle$.(3pts)

b) Symmetric Property: $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{v}, \boldsymbol{u}\rangle$.(3pts)

c) Positive Definite Property: For any $\boldsymbol{u}$, $\langle \boldsymbol{u}, \boldsymbol{u}\rangle \geq 0$; and $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 0$ if and only if $\boldsymbol{u} = 0$.(4pts)

2. Verify that the norm defined in the lecture is indeed a "norm", that is, it satisfies the following(10pts)

N1: $\|\boldsymbol{v}\| \geq 0$ and $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = 0$.(3pts)

N2: $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$.(3pts)

N3: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|u\| + \|v\|$(4pts)

3.Let $u$ and $v$ be vectors with dimension $n$. Prove the following identity (15pts):

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

4. *Symptoms vector*. A vector $s$ records whether each of 20 different symptoms is present in a medical patient, with $s_i = 1$ meaning the patient has the symptom and $s_i = 0$ meaning he does not. Express the following using vector notation. (10pts)

a) Express "The total number of symptoms the patient has." (5pts)

b) Express "The patient exhibits five out of the first ten symptoms." (5pts)

Remark: For example, the patient exhibits the 1st, 2nd, 5th, 8th, 10th, and 16th symptons, but not other symptons. As another example, the patient exhibits the 2nd, 6th, 7th, 8th, 10th, 13th, and 15th symptons but not other symptons.

Hint: (a) Define a vector, and use an inner product. (b) Use an equation.

5. *Interpretation of angles.* Denote $\angle(\mathbf{u}, \mathbf{v})$ as the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. Suppose that there are three types of fruit in the supermarket: orange, apple, and banana. Every week, Alice buys 1 kilos of oranges, 3 kilos of apples and 5 kilos of bananas; Bob buys 2 kilos of oranges, 6 kilos of apples and 10 kilos of bananas; Charlie buys 5 kilos of oranges, 1 kilos of apples and 0 kilos of bananas. The three vectors corresponding to their choices are $\mathbf{a} = (1, 3, 5)$, $\mathbf{b} = (2, 6, 10)$, $\mathbf{c} = (5, 1, 0)$. (15pts)

(i) Compute the two angles $\theta_1 = \angle(\mathbf{a}, \mathbf{b})$ and $\theta_2 = \angle(\mathbf{a}, \mathbf{c})$. (5pts)

Hint: You can use the website Wolfram Alpha to compute the arccos function; see `https://www.wolframalpha.com/input?i=arc+cos+0`.

(ii) Which one is larger? Provide an interpretation of this comparison.(5pts for correct interpretation, a correct guess without interpretation will receive 0)

Hint: The keyword you can use is "preference" or "taste". Use your own words to describe the interpretation. The answer can be in a few sentences.

(iii) In general, when evaluating an object (e.g. a university, a restaurant, a cellphone), everyone has their own "weight vector". Provide an interpretation of the angle between the weight vectors of two people. (5pts)

6.*Interpolation of rational functions*. A rational function of degree two has the form

$$f(t) = \frac{c_1 + c_2 t + c_3 t^2}{1 + d_1 t + d_2 t^2}$$

where $c_1, c_2, c_3, d_1, d_2$ are coefficients. ('Rational' refers to the fact that $f$ is a ratio of polynomials. Another name for $f$ is bi-quadratic.) Consider the interpolation conditions

$$f(t_i) = y_i, i = 1, \ldots, K$$

where $t_i$ and $y_i$ are given numbers. Express the interpolation conditions as a set of linear equations in the vector of coefficients $\theta = (c_1, c_2, c_3, d_1, d_2)$, as $A\theta = b$. Give $A$ and $b$, and their dimensions.(20pts)

7. *Solve the system of linear equations.* Solve the following linear systems using Gaussian elimination. (20pts, 10 for each system)

Remark: You shall represent the systems with augmented matrices first. If you solved the system, but did not use augmented matrices, you would only get $1/3$ of the points.

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 = 7 \\ 2x_1 + x_2 + 3x_3 = 6 \\ x_1 + 2x_2 + x_3 = 4 \end{cases} \tag{1}$$

$$\begin{cases} 5x_1 + 4x_2 + x_3 = 16 \\ x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 2x_2 + x_3 = 9 \end{cases} \tag{2}$$