Lecture 01

Introduction to Linear Algebra and Data Science

香港中文大學(深圳) | 数据科学学院
The Chinese University of Hong Kong, Shenzhen | School of Data Science

Why Data are Important

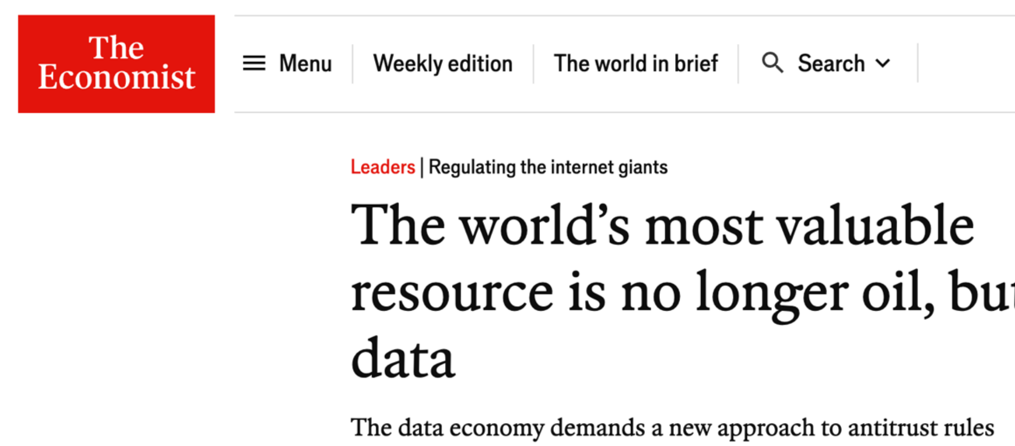
Data are the new "oil"! 数据是新的石油

Oil can drive cars, airplanes, cellphones, …

Hardware

Data can drive AI and decision making,

"Software"


The Economist
Menu
Weekly edition
The world in brief
Search

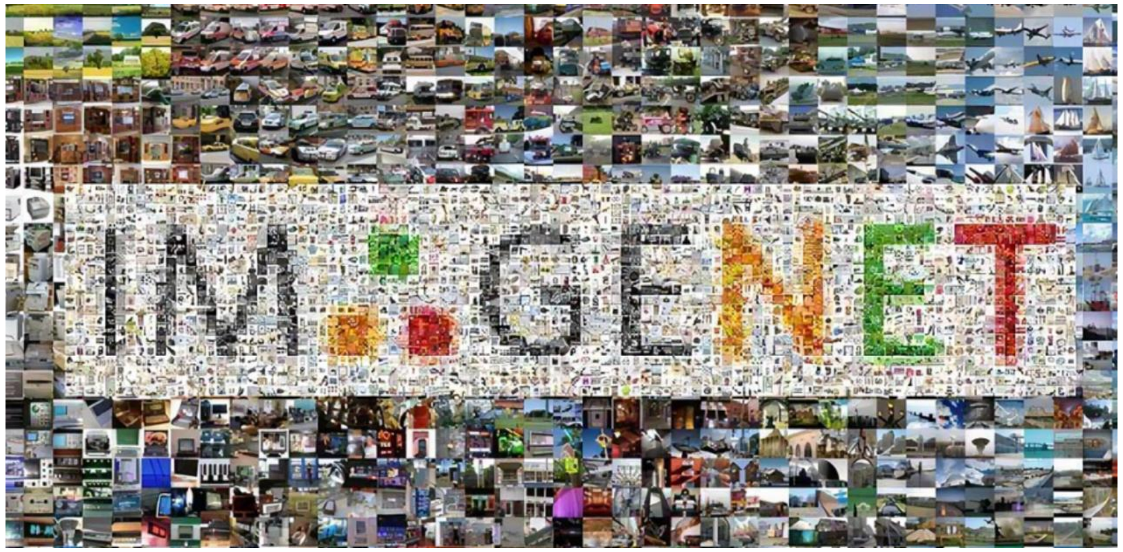
Leaders | Regulating the internet giants

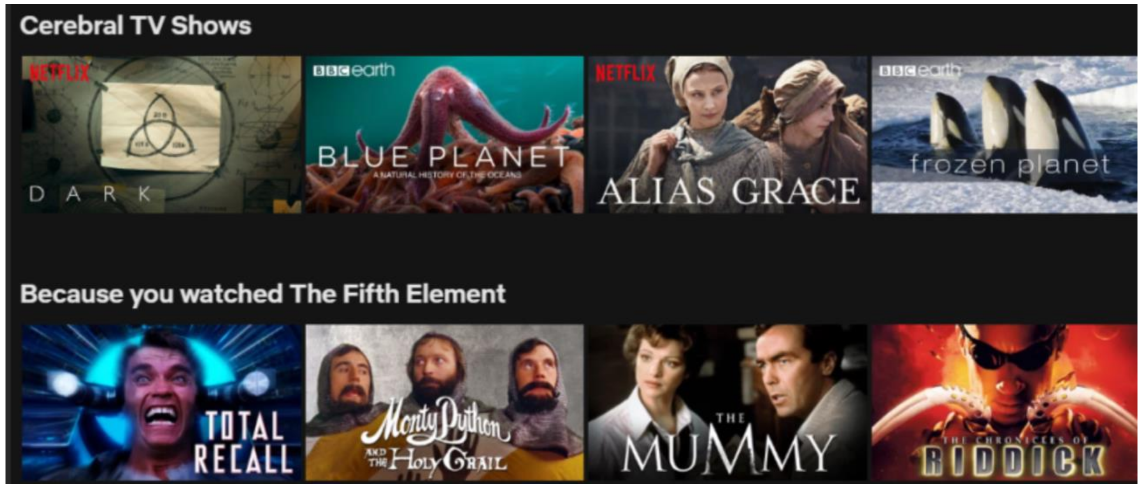
The world's most valuable resource is no longer oil, but data

The data economy demands a new approach to antitrust rules

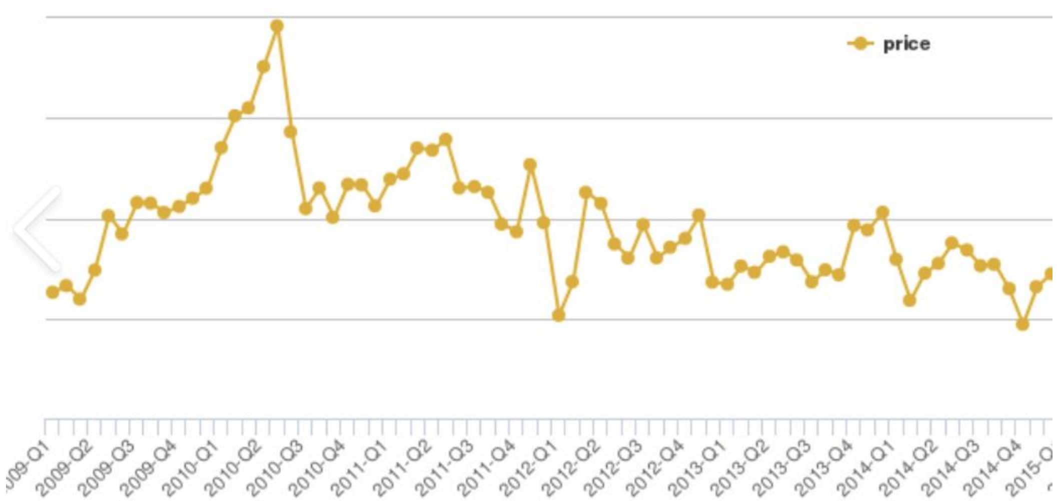
2017, Economist article

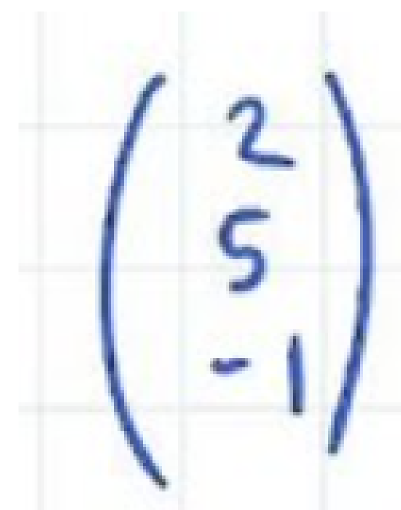
What are “Data”?





First Challenge: How to “Express” them?

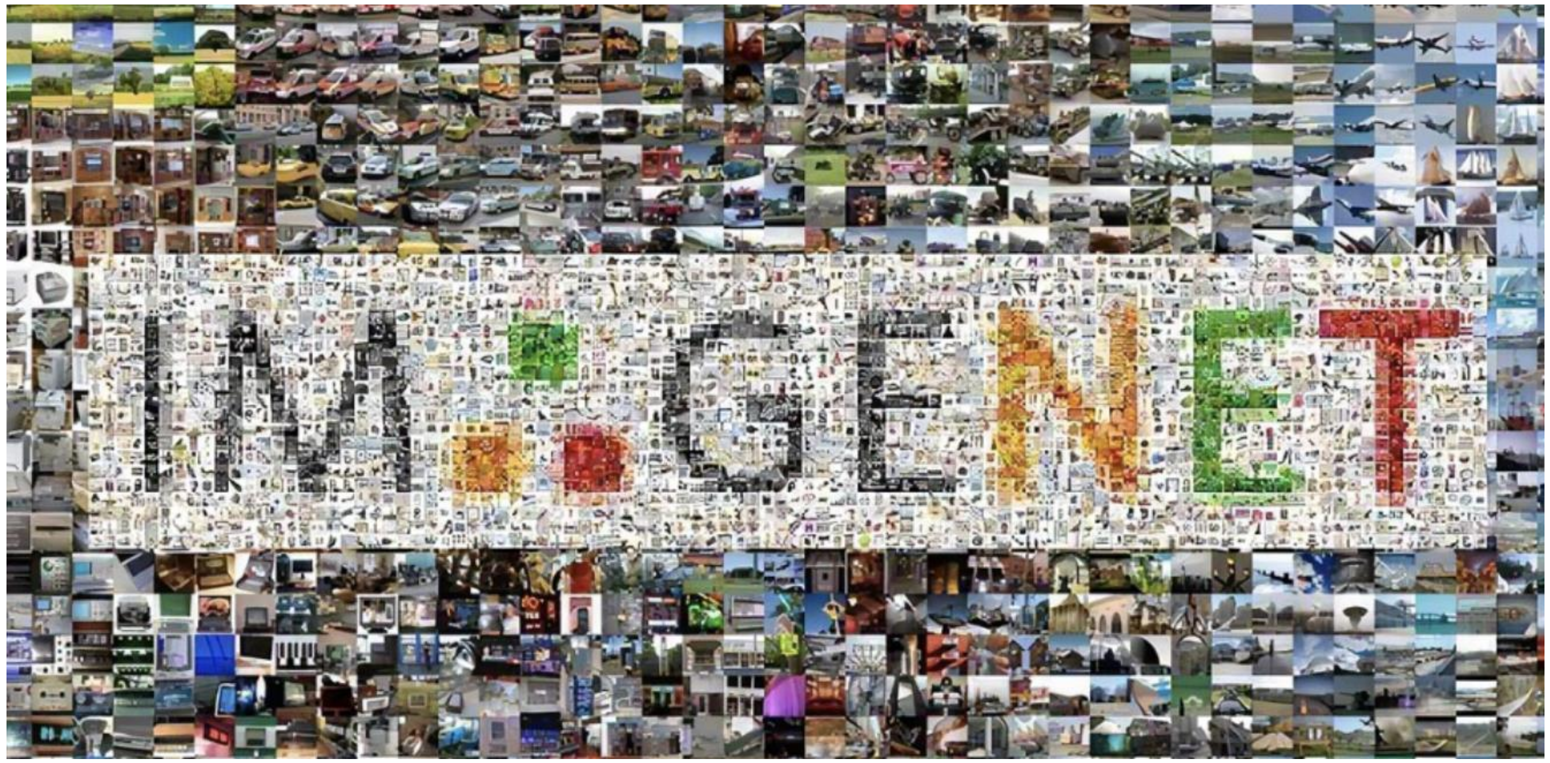




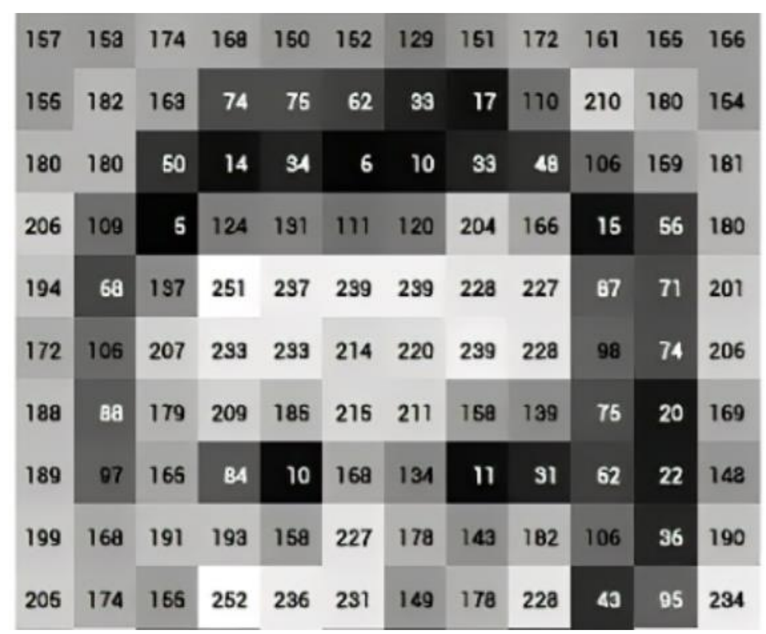
Figure?

vector!

First Challenge: How to “Express” them?



Figure?



Matrix!

What is Linear Algebra?

Algebra: Mathematical representation of problems and operations

X, Y, x, y, A, B, …

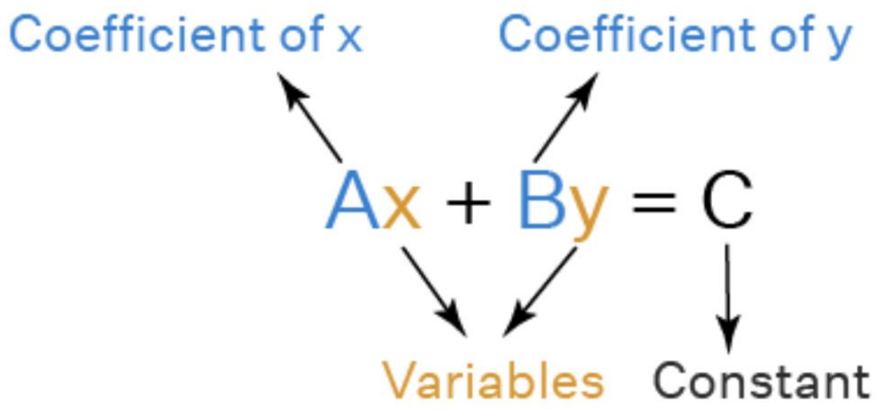
addition +
subtraction -
multiplication ×
inner product ⟨·,·⟩ …

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra



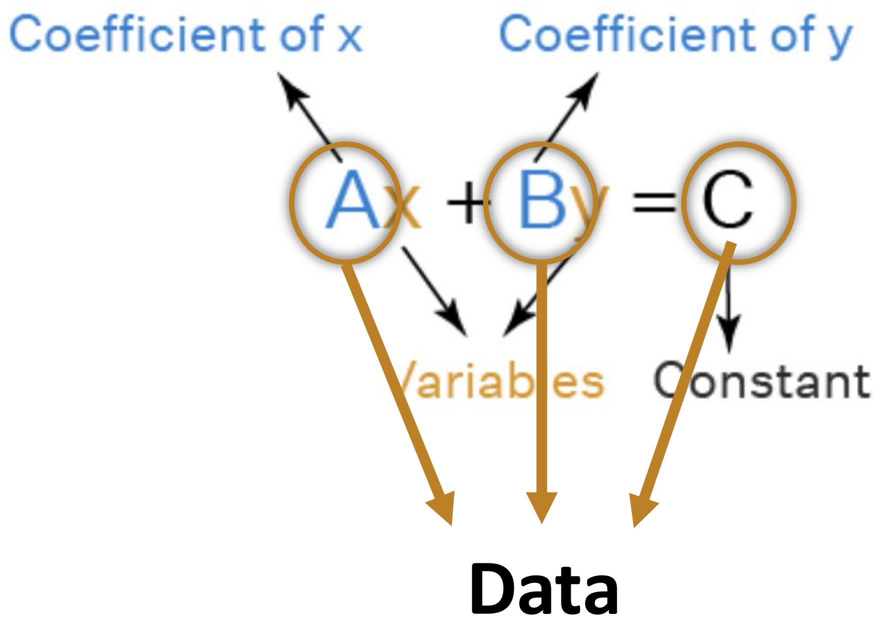
Standard form of an Linear Equation

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra



Standard form of an Linear Equation

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra

$$A\mathbf{u} + B\mathbf{v} = \mathbf{c}$$

Matrices of **Data**

Standard form of a **System** of Linear Equations

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra

$$A\mathbf{u} + B\mathbf{v} = \mathbf{c}$$

Vectors of Parameters/Data

Standard form of a **System** of Linear Equations

What is Linear Algebra?

Algebra: Mathematical representation of problems and operations

X, Y, x, y, A, B, …

addition +
subtraction -
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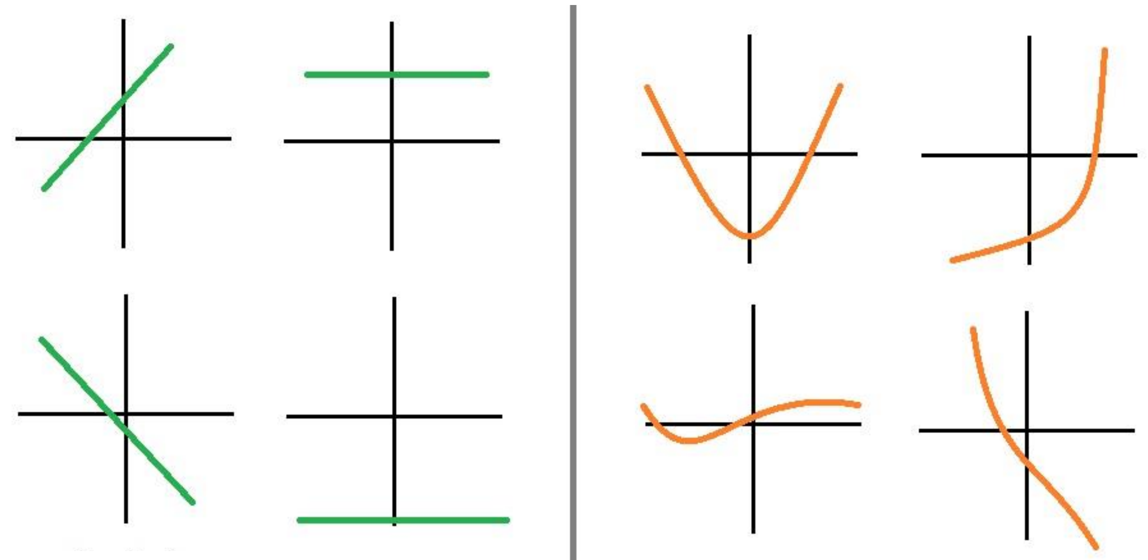
inner product ⟨·,·⟩ …

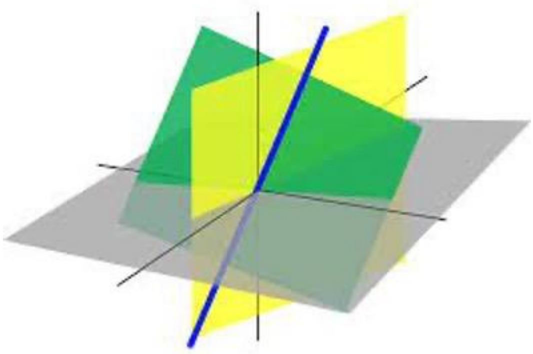
Linearity:



linear functions **non-linear functions**

2-D representation

What is Linear Algebra?



Mathematically, **vector spaces**

and linear transformations

Applications

- **Machine learning and data science**
- Computer vision and graphics
- Graph theory
- Control theory
- Cryptography
- Fractals and chaos
- Energy systems
- Network systems
- Genetics
- Etc. …

Practically, it can be applied to **any** problems with

vector, matrix-type data, and linear models

Why Need Linear Algebra? A1: Fundamental

Quora
Search for questions, people, and topics

What exactly is linear algebra? Why do we need it?

Why study linear algebra?

Ask Q

Asked 9 years, 8 months ago Modified 3 years, 4 months ago Viewed 133k times

128

Simply as the title says. I've done some research, but still haven't arrived at an answer I am satisfied with. I know the answer varies in different fields, but in general, why would someone study linear algebra?

https://math.stackexchange.com/questions/256682/why-study-linear-algebra

Linear algebra is beyond important, it is fundamental to so many fields that I cannot count them all.

Having studied Engineering, I can tell you that Linear Algebra is fundamental and an extremely powerful tool in **every single** discipline of Engineering.

3D computer graphics? Linear algebra.
Quantum mechanics? Linear algebra.
Weather forecast models? Linear algebra.

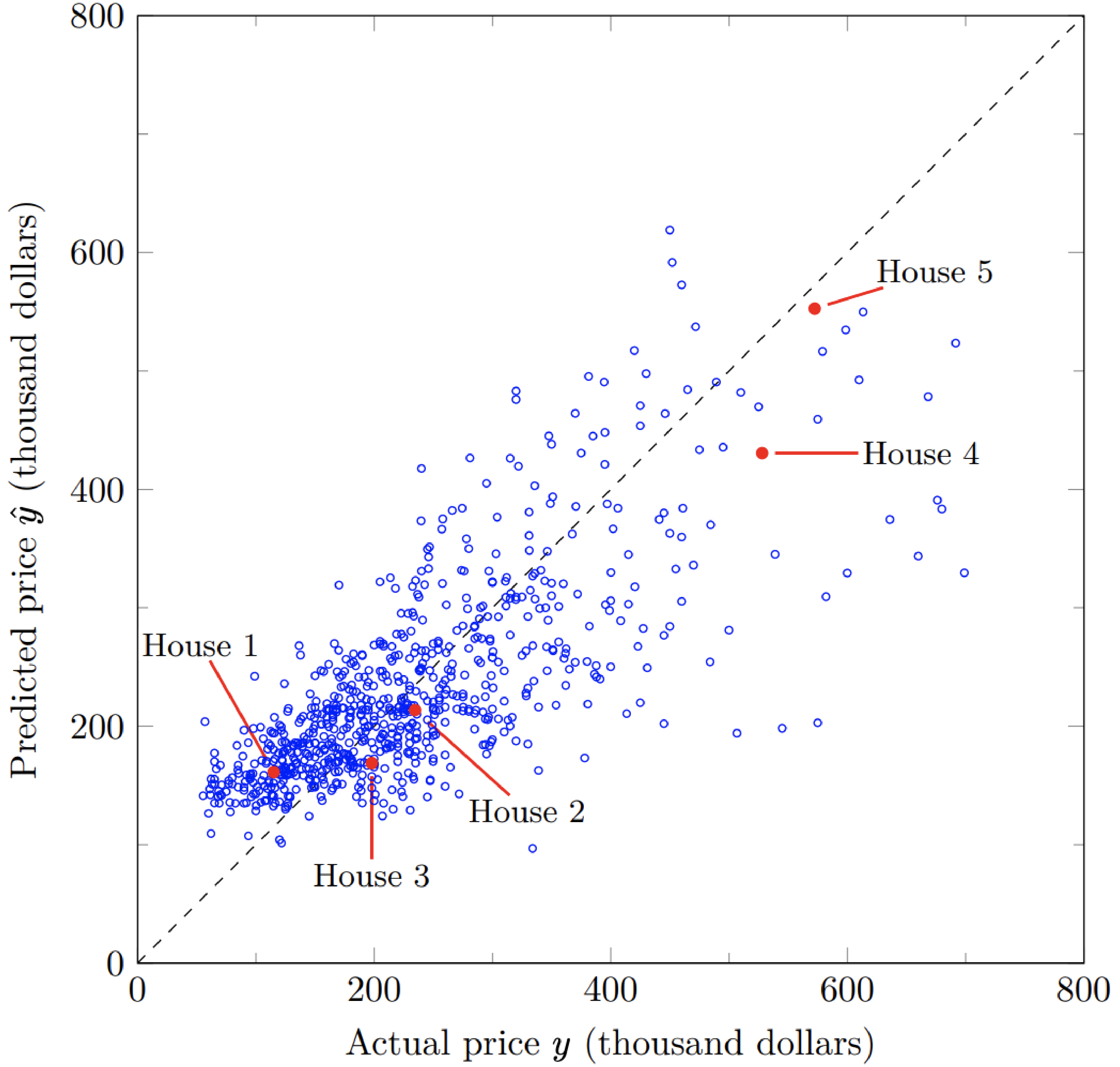
Study it if you are into economics, computer graphics, physics, chemistry, statistics or anything quantitative (in today's world, that's everything).

Why Learning Linear Algebra is Critical?

- It can be applied to **many** problems
- **Easy** to model, analyze, and compute (not an easy subject)
- Foundations of more advanced and complex methods

Linear Models	Nonlinear Models
Less accurate	More accurate
Easy	Hard

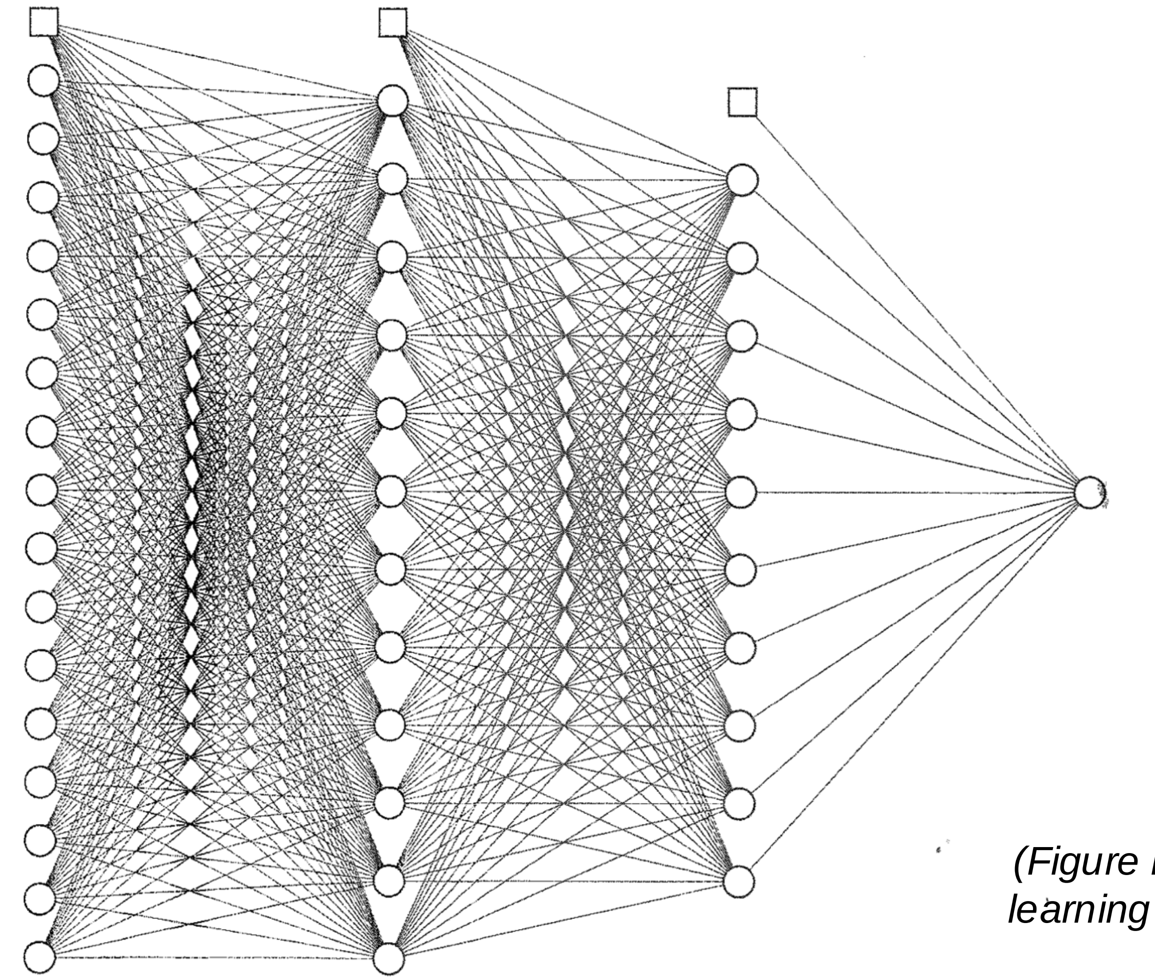
Why Learning Linear Algebra is Critical?



Actual and predicted sale prices of houses in Sacramento during 5 days

Example

Convolutional Neural Networks (CNNs)

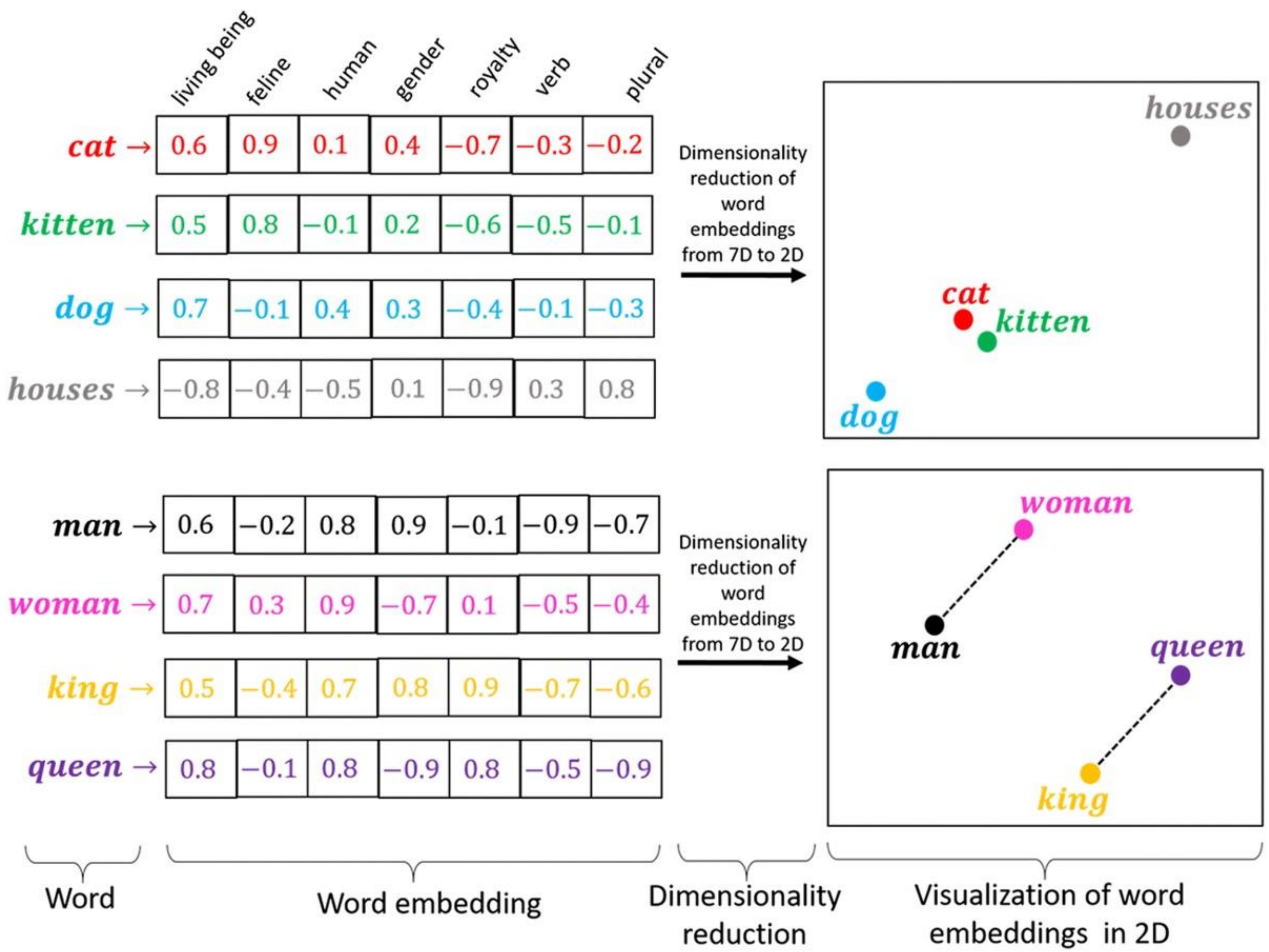


(Figure resource: linear algebra and learning from data by Gilbert Strang)

- Each diagonal is a weight to be learned by optimization
- Edges from the square contain bias **vectors** and the other weights are shared in **matrices**

Example

Word Embedding: transform words to vectors



(Figure resource: https://medium.com/@hari4om/word-embedding-d816f643140)

This Course

It is a math course.

Mathematical definitions, lemmas, theorems, and formal expressions will be marked in a colored box..

But not a “pure math”-type course

Skipped some abstract math results compared to a math course

More data science problems will be demonstrated to motivate the concepts (compared to classical Linear Algebra courses)

For instance, regression problems, graph matrices, searching, etc. (will cover as much as possible)

--This is strongly advocated by Steven Boyd’s book and course in Stanford

Strategies for Effective Math Learning: General

Practice Regularly: Math is learned by doing.

--Solve as many practice problems in the books you can find

Review Consistently

--formal review: biweekly or weekly

--informal review:

just recall what you learned, during lunch or walking

Collabarative Learning

Discuss with Peers:

Encourage forming study groups to share understanding and problem-solving strategies.

- Teach peers

  "Teaching others helps reinforce your own understanding."
-

- Ask and answer questions on Plazza

Active Learning

Draw mind-map & write summary yourself.

--Ask yourself: what did I really learn?

Link Math to Real-Life Applications

--I gave examples of real-life applications, you can try to identify more

Refer to Multiple Texts

--Every author explains concepts differently—explore what resonates with you.

General Goal

Classic linear algebra training:

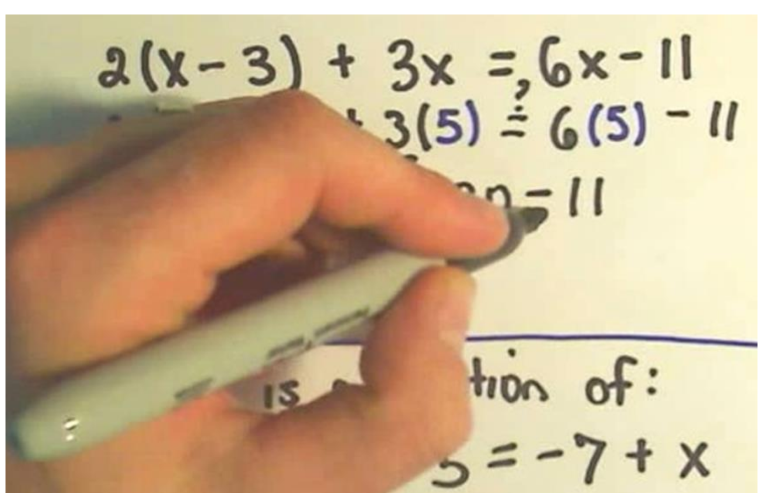
How to solve linear equations as fast as possible as humans?

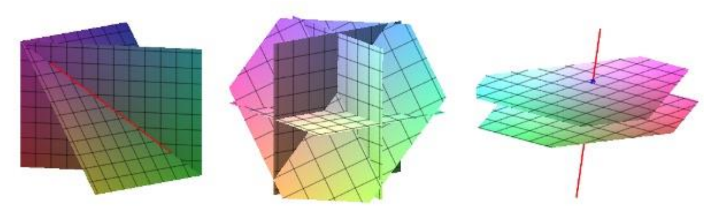
Computers will do the job!

(Unfortunately, in your exams, you sometime need to solve linear equations by hands; just occasionally!)

Understand Concepts, Not Just Procedures

Learn **How** and **what,** and also **Why (**keep asking yourself why





Basic Components in Linear Algebra

Question1: What can you think about the basic components in Linear Algebra from your high school knowledge?

Vectors, matrices, and their operations etc

Question2: What are the most fundamental component in Linear Algebra?

Vectors!

Examples of Vectors

"The world is continuous, but the mind is discrete"

- David Mumford

How to interpret ?

Vectors

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]$$

(row) vector — $\mathbf{v}$

entry/element — v_i

Vectors

Definition (column vectors)

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^T$$

Transpose

column vector

entry/element

$$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \equiv (v_1, \ldots, v_n)$$

also written as

Convention: vectors considered as columns

Examples of Vectors

Example:

Zero Vector

$$\mathbf{v} = (0,0,0,\ldots,0)$$

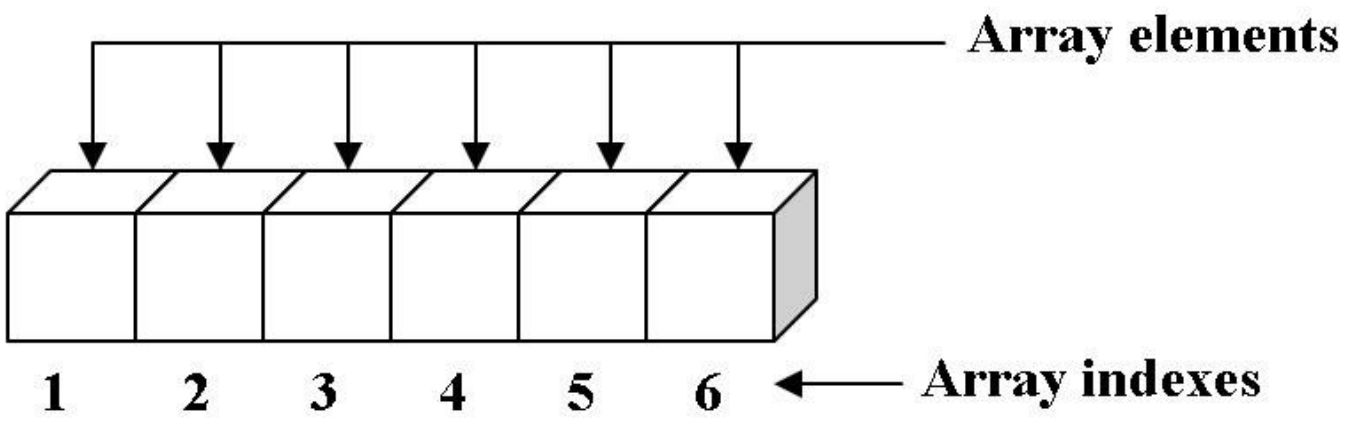
One Vector

$$\mathbf{v} = (1,1,1,\ldots,1)$$

Examples of Vectors

Example:

An **array** data structure in computer algorithms

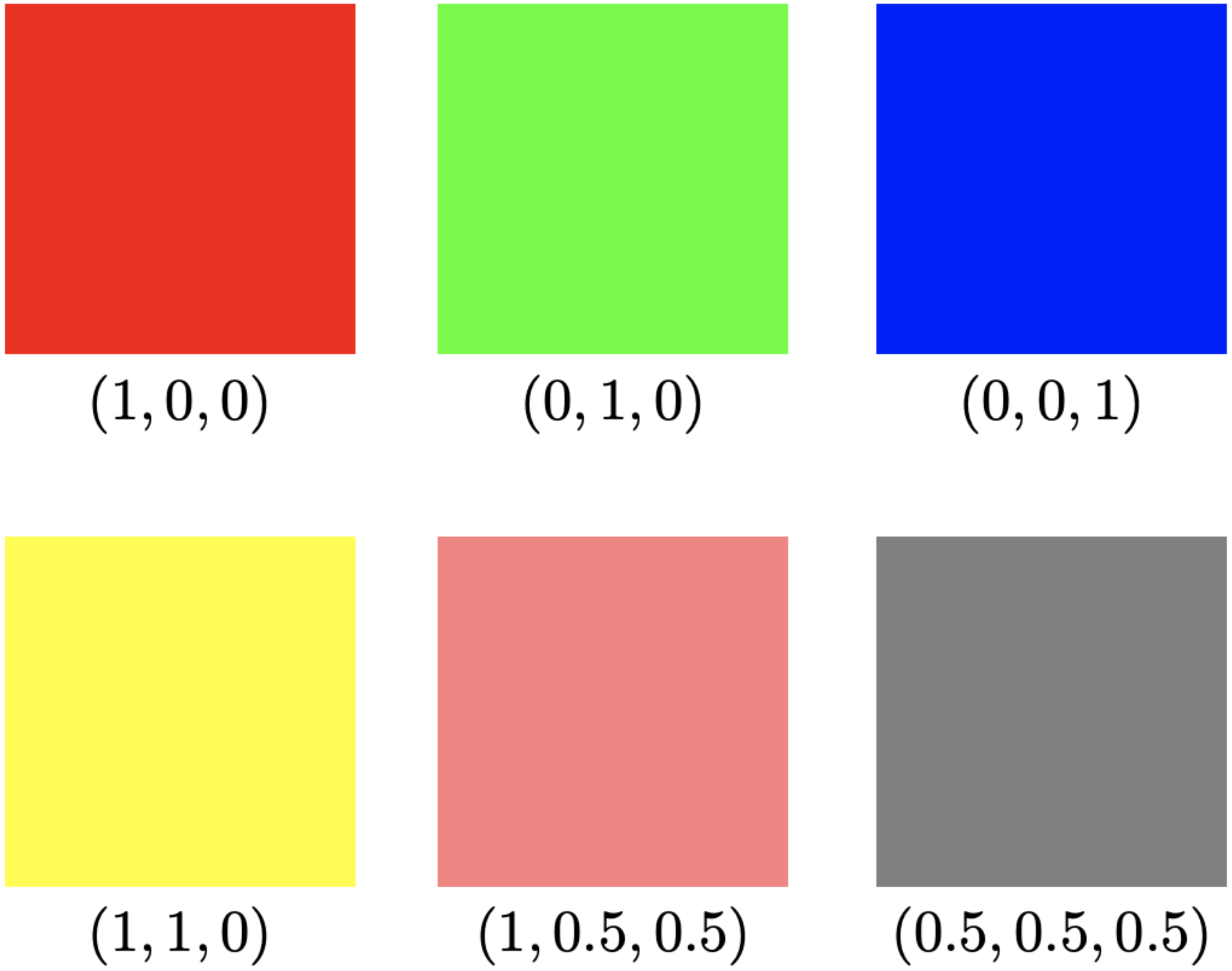


One-dimensional array with six elements

Examples of Vectors

Example:

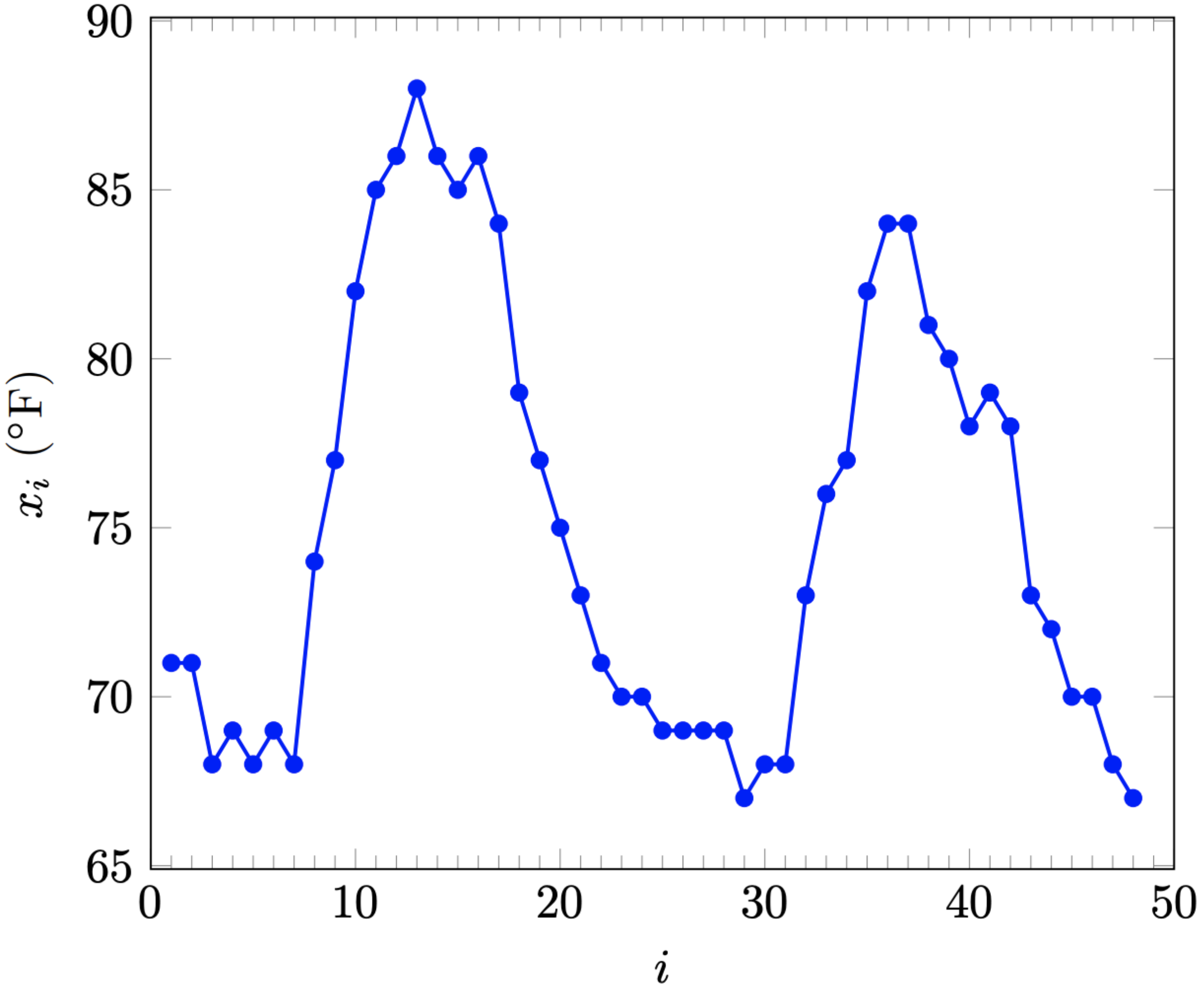
Color RGB Vectors



Examples of Vectors

Example:

A **time series** (a sequence of data points) in data science problems

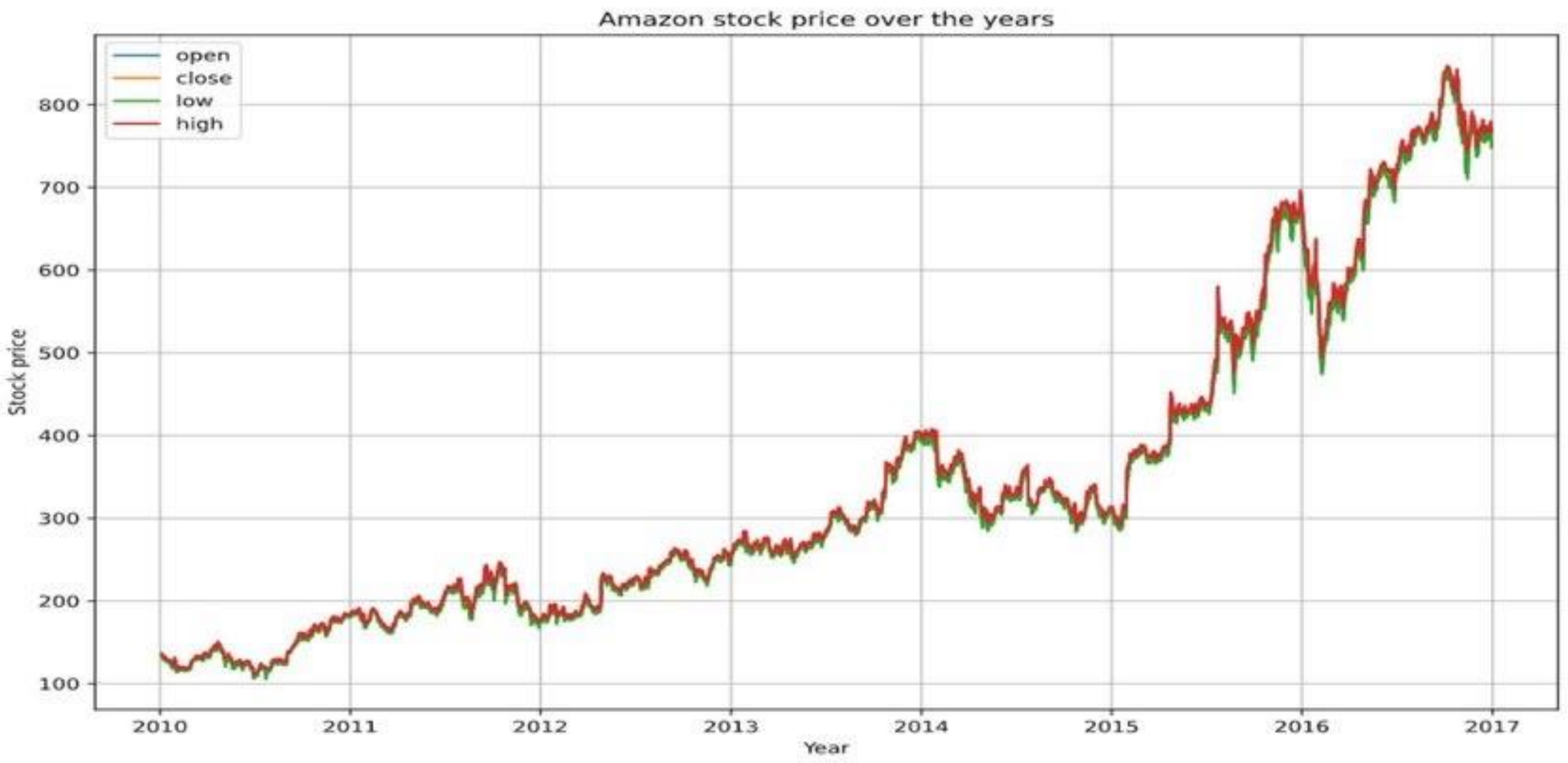


Hourly Temperature in LA on Aug 5 and 6, 2015

Examples of Vectors

Example:

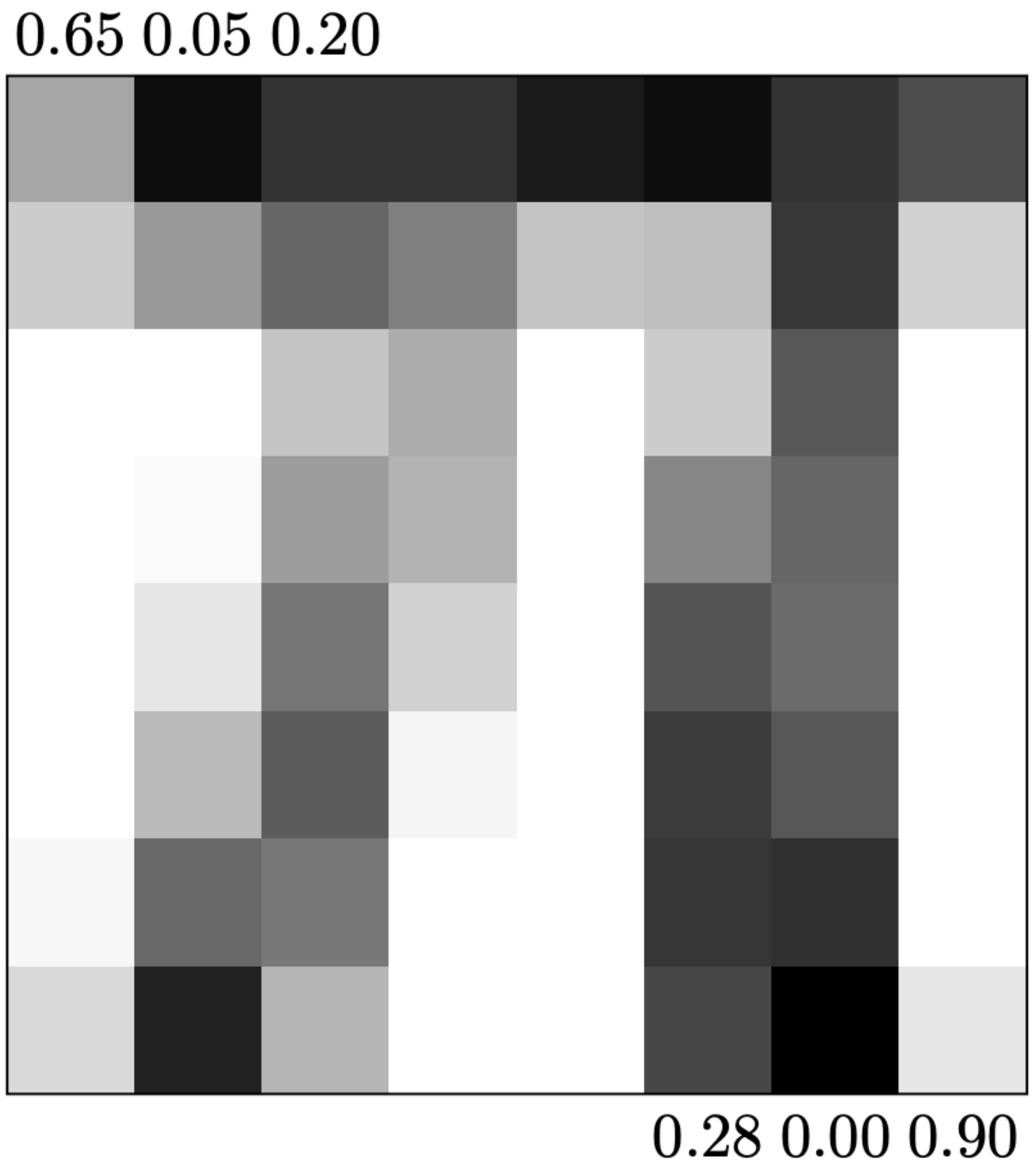
(In Finance) Stock Prices



Examples of Vectors

Example:

Images



Vectors

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^{\mathrm{T}}$$

T → Transpose

$\mathbf{v}$ → column vector

v_i → entry/element

Example:

A **list** data structure in computer algorithms

A time series (a sequence of data points) in data science problems

Vector Operations

Vector Addition

$$\mathbf{v} = (v_1, \ldots, v_n)$$

$$\mathbf{w} = (w_1, \ldots, w_n)$$

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, \ldots, v_n + w_n)$$

Element-wise Operations!

Vector Multiplication

$$\mathbf{v} = (v_1, \ldots, v_n)$$

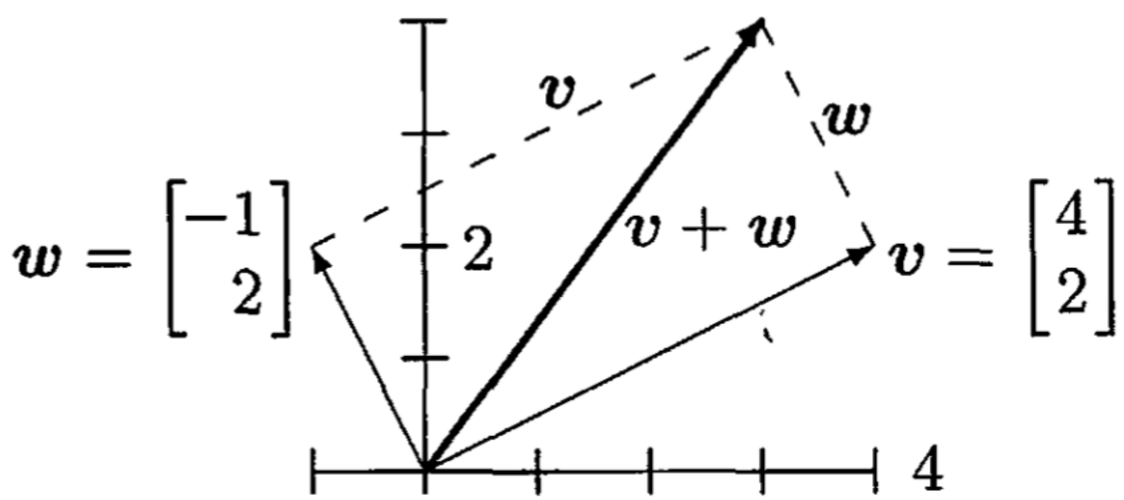
$$c\mathbf{v} = (cv_1, \ldots, cv_n)$$

Remark: *We do not worry about the domain of the elements* $v_1, \ldots, v_n$ *and* c *so far. They can be chosen from the set of real/complex numbers*

Visualization of Vector Operations

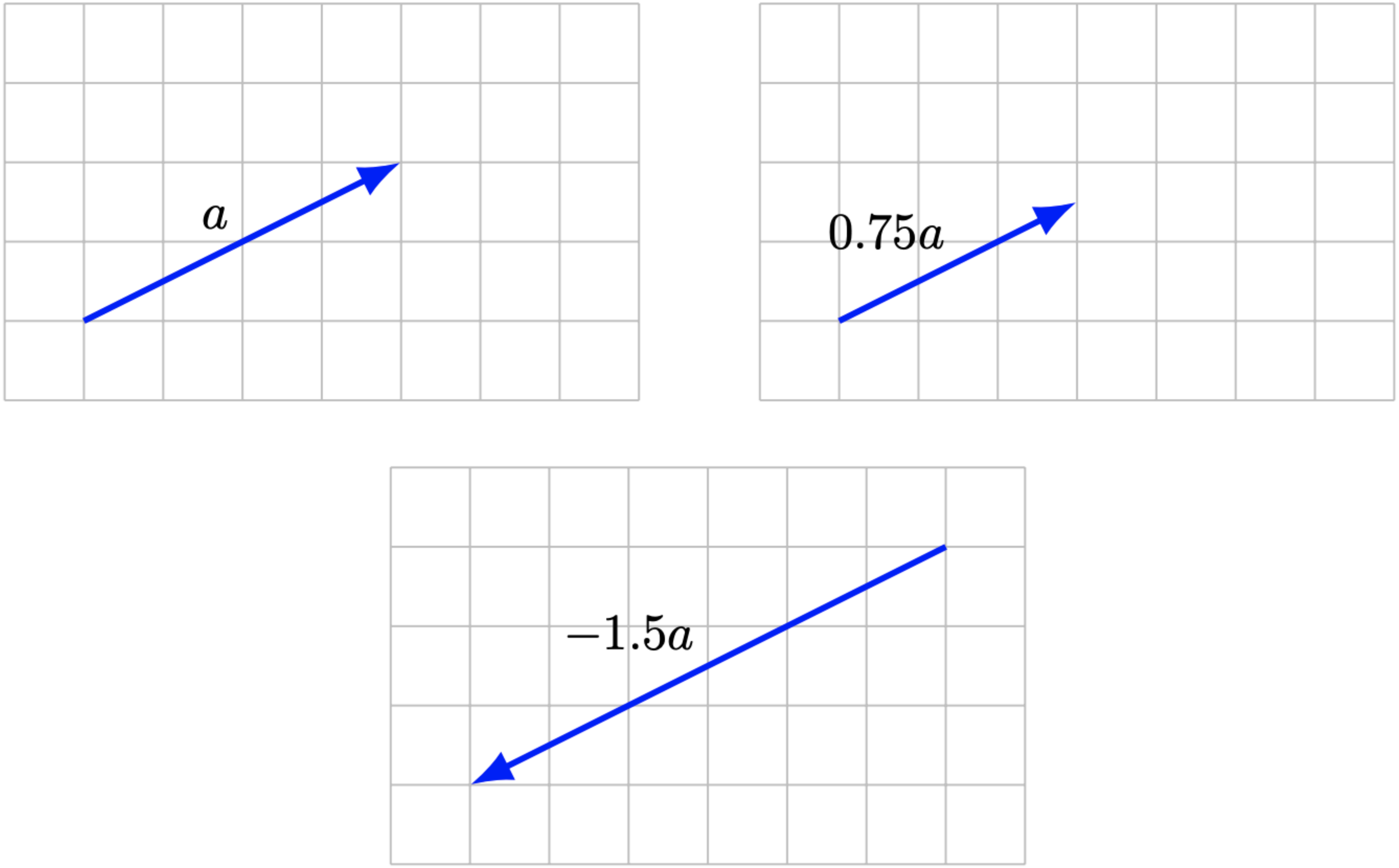
Addition $\quad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Visualization of Vector Operations

Multiplication (by a scalar)



Vector Operations

Linear Combination

$$\mathbf{v} = (v_1, \ldots, v_n)$$

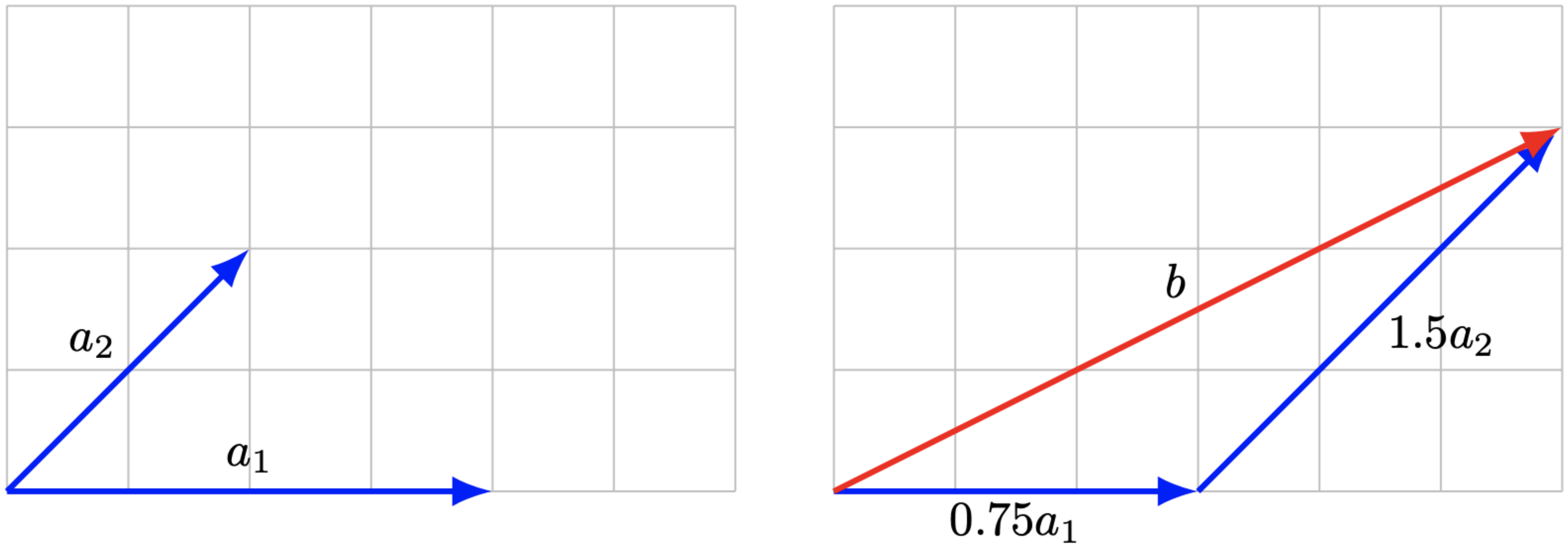
$$\mathbf{w} = (w_1, \ldots, w_n)$$

$$c\mathbf{v} + d\mathbf{w} = (cv_1 + dw_1, \ldots, cv_n + dw_n)$$

We only need "**addition**" and "**multiplication**" to derive the "**linear combination**" law

Visualization of Linear Combination

Linear Combination

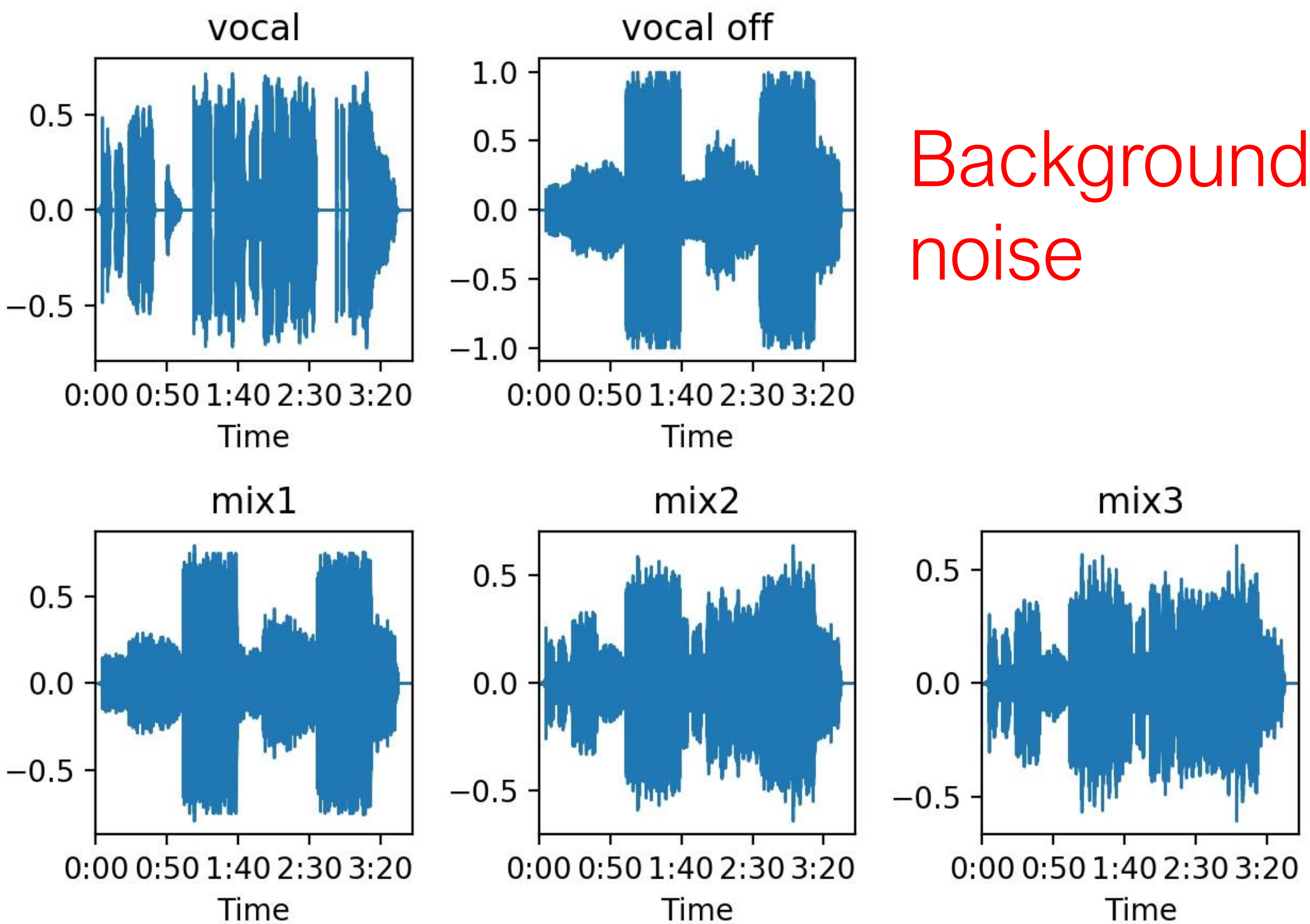


Examples of Linear Combination

Linear Combination

Example: **Video/audio mixing**

Human Voice



Background noise

More Examples

Example (Combinations that are not linear)

$$\mathbf{u} = (u_1, u_2)$$

$$\mathbf{v} = (v_1, v_2)$$

$$f(\mathbf{u}, \mathbf{v}) = (u_1 v_1, u_2 + v_2)$$

Combination, but NOT linear combination!

More Examples

Example (Linear Combination of Vectors)

What combination $c\mathbf{v} + d\mathbf{w}$ produces $\mathbf{u}$?

$$\mathbf{v} = (1,2)$$

$$\mathbf{w} = (3,1)$$

$$\mathbf{u} = (14,8)$$

Answer:

$$c = 2$$

$$d = 4$$

More Examples

Example (Vector Operations)

Find $\mathbf{v}$ and $\mathbf{u}$ such that

$$\mathbf{u} + \mathbf{v} = (4,5,6)$$

$$\mathbf{u} - \mathbf{v} = (2,5,8)$$

Answer:

$$\mathbf{u} = (3,5,7)$$

$$\mathbf{v} = (1,0,-1)$$

Summary

In today's lecture, we have covered

- Definition of **vectors**
- Vector Operations

(Textbook Section 1.1)

(Slides/Notes can be found in our course webpage)

Vector Operations

Question: Can you think about any other vector operations?

The next lecture!

A General Question: What if we have three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$

Can we write $\mathbf{x}$ as a linear combination of $\mathbf{y}, \mathbf{z}$?

The next next ⋯next lecture!