Lecture 02

Vector II: Norm and Inner Product

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Recall

In the last lecture …

- Definition of **(column) vectors**
- Basic vector operations (**addition, multiplication, linear combination**)

Linear Algebra Terminology and Conventions

- A **vector** is an ordered list of numbers (often a column) written as

$$\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \quad \text{or} \quad (-1.1, 0, 3.6, -7.2)$$

- Numbers in the list/array are the elements (entries, coefficients, components)
- Number of elements is the size (dimension, length) of the vector (n-length) (n-dimensional)

  [*Warning*]: In some textbooks, like Gilbert Strang, length refers to something else
- The vector above has dimension 4; its third entry is 3.6
- A vector of size n is also called an n-vector
- Numbers are called scalars (compared to vectors)

Linear Algebra Terminology and Conventions

- Vectors are often represented by mathematical symbols, e.g.,

$$x, y, u, v$$

It could be either in boldface or not, depending on contexts

Other conventions for beginners and engineers: $\mathbf{u}, \mathbf{v}$ or $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}$

- The i-th element of an n-vector $\mathbf{u}$ is denoted as u_i
- In u_i, i is the index
- For an n-vector, indexes run from $i = 1$ to $i = n$

[*Warning*]: sometime u_i may be used to denote the i-th vector in a list of vectors

- Two vectors $\mathbf{u}, \mathbf{v}$ are equal iff $\mathbf{u} - \mathbf{v} = 0$, written as $\mathbf{u} = \mathbf{v}$

Linear Algebra Terminology and Conventions

- Zero vector: All elements are zeros, e.g., $(0,0,0,0)$ denoted by 0, $\mathbf{0}$, or $\mathbf{0}_n$
- One vector: All elements are ones, e.g., $(1,1,1,1)$ denoted by 1, $\mathbf{1}$, or $\mathbf{1}_n$
- Non-zero vector: Not a zero vector
- Non-one vector? Rarely seen!

Learning Goals Today

Today: More vector operations (1) Vector Norm; (2) Inner Product.

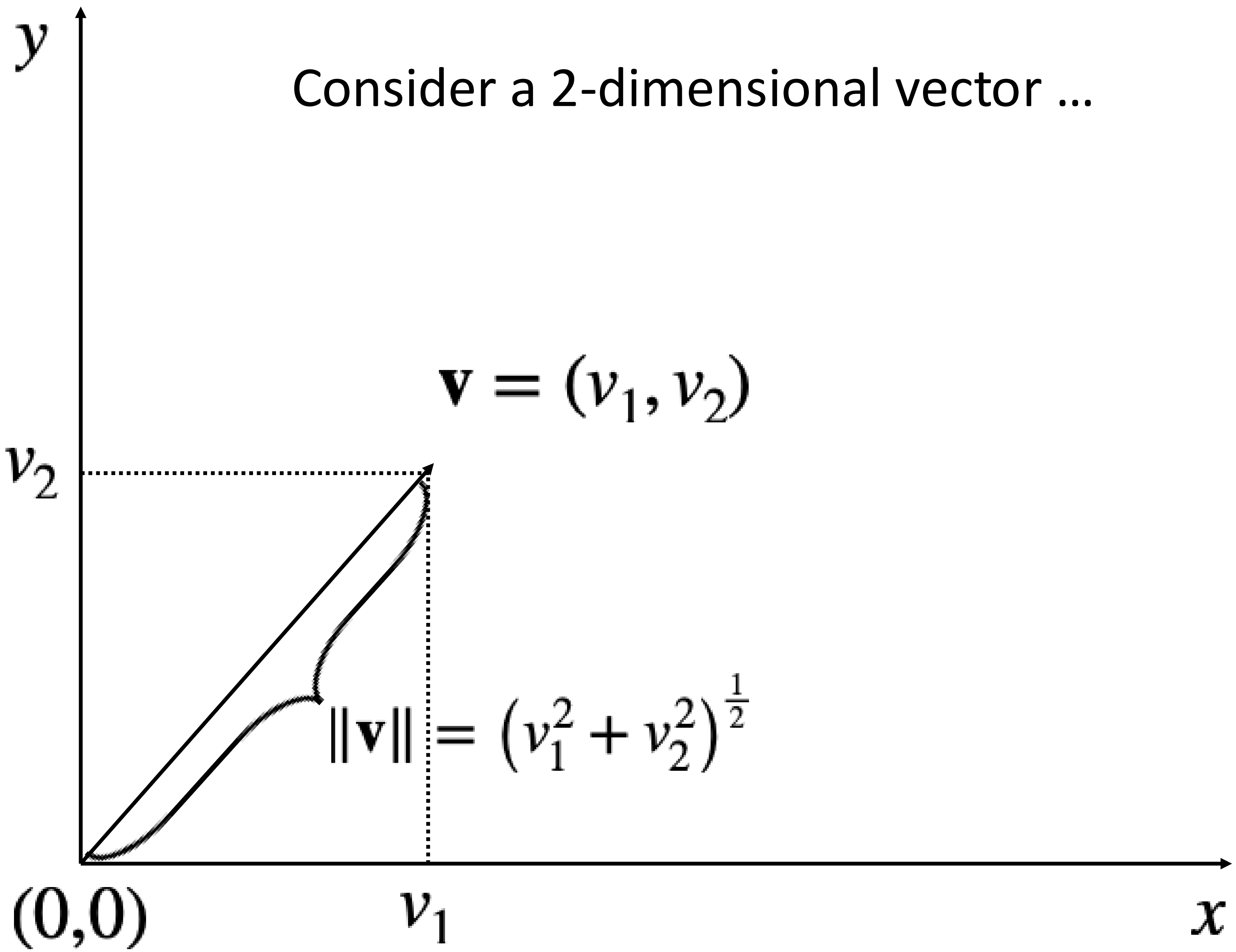
After this lecture, you should be able to:

- calculate the **norm & inner product**
- tell **a few major properties of** norm & inner product
- utilize the **Cauchy Schwarz inequality** and **triangular inequality**
- provide real-world applications of "**inner product**”

Part I Vector Norm

Vector Norm ("Geometric Length")

Consider a 2-dimensional vector …



Vector Norm

A generalized notion of "*absolute value*" …

Definition (ℓ_2 -norm)

Let $\mathbf{v} = (v_1, \ldots, v_n)$ be a n-length vector. The ℓ_2**-norm** of $\mathbf{v}$, denoted by $\|\mathbf{v}\|_2$ is defined as

$$\|\mathbf{v}\|_2 := \left(v_1^2 + \cdots + v_n^2\right)^{\frac{1}{2}}$$

The ℓ_2-norm is also called an *Euclidean norm* or geometric length

We often abbreviate $\| \cdot \|_2$ as $\| \cdot \|$

Properties of Vector Norms

Properties of Vector Norms

P1 $\|\mathbf{v}\| \geq 0$ $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = 0$

P2 $\|c\mathbf{v}\| = ?$

P3 $\|\mathbf{u} + \mathbf{v}\|$? $\|\mathbf{u}\| + \|\mathbf{v}\|$

Unit Vector

Definition (Unit Vector)

A vector $\mathbf{v}$ is called a **unit vector** if $\|\mathbf{v}\| = 1$

For any non-zero vector $\mathbf{v}$, $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector

Unit Vector: Examples

Examples (Unit Vector)

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Standard Unit Vectors in a Cartesian coordinate system

[*Warning*]: In some textbooks, like Stephen Boyd, only above vectors are called unit vectors

Part II Inner Product

Vector Operations

Question1: Can you think about any other vector operations?

Vector Addition

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

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How about
$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_2 w_2 \\ \vdots \\ v_n w_n \end{bmatrix}$$
?

This is a “point-wise product”.

Dot Product or inner product

Definition (Dot Product)

A *dot product* between two vectors (of the same size)

$$\mathbf{v} = (v_1, \ldots, v_n)$$

$$\mathbf{w} = (w_1, \ldots, w_n)$$

is defined as

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i$$

Example: Calculate the dot product of (-1,2,2) and (1,0,-3)

Dot Product or Inner Product

Sometime we also write a dot product as $\mathbf{v} \cdot \mathbf{w}$ or $\mathbf{v}^\top \mathbf{w}$

Recommend writing as $\langle \mathbf{v}, \mathbf{w} \rangle$ to avoid confusion.

A *dot product* is called an *inner product* in more general settings

Properties of Inner Products

Properties of Inner Products

P1 Linearity $\langle a\mathbf{v} + b\mathbf{u}, \mathbf{w}\rangle = a\langle \mathbf{v}, \mathbf{w}\rangle + b\langle \mathbf{u}, \mathbf{w}\rangle$ for all $\mathbf{u}$ $\mathbf{v}$ $\mathbf{w}$

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P2 Symmetry $\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$ for all $\mathbf{v}$ $\mathbf{w}$

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P3 Positivity $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ for all $\mathbf{v}$

$\langle \mathbf{v}, \mathbf{v}\rangle = 0$ iff $\mathbf{v} = 0$

Properties of Inner Products

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$\langle \mathbf{v}, \mathbf{v}\rangle = 0$ iff $\mathbf{v} = 0$

verify $\langle \mathbf{v}, \mathbf{v}\rangle = \|v\|^2$!

Reading: Inner product naturally induces a norm and every “inner product space” is a “normed vector space”

Application 1: Weight, Feature and Score

Example 1.1: My MAT2041 final score

	Assignment &Quiz & Attendence	Midterm	Final	Total
Weight	0.35	0.3	0.35	
Score	90	85	95	
Cost	31.5	25.5	33.25	90.25

"weight vector"

"Feature vector"

"Score"

$\mathbf{w}$ = (0.35, 0.3, 0.35): weight vector

$\mathbf{v}$ = (90, 85, 95): feature vector

$\langle \mathbf{v}, \mathbf{w} \rangle$ = 90.25 =: score

Application 1: Weight, Feature and Score

Example 1.2: Movie preference

	Action Film	Hollywood	Comedy	Total
Weight	10%	10%	80%	
Value	0.5	0.5	10	
Score	0.05	0.05	8	8.1

“weight”

“Feature”

Preference of Adam

Movie 1: Moon Man

“Score”

Application 1: Weight, Feature and Score

Example 1.2: Movie preference

Movie 2: The Matrix

	Action Film	Hollywood	Comedy	Total
Weight	10%	10%	80%	
Value	10	8	2	
Score	7	1.6	0.2	8.8

"weight"

"Feature"

Preference of Adam

Movie 2: Matrix

"Score"

Application 1: Weight, Feature and Score

Application 1: (general)

$\mathbf{w}$ is a vector of the same size (often called a weight vector),

$\mathbf{v}$ represents a set of "features" of an object,

Score: inner product $\langle \mathbf{v}, \mathbf{w} \rangle$ is a weighted sum of the feature values.

Examples:

	Grading	**Movie**	**Your example?**
"Weight"	Grading Scheme	Personal Preference	?
"Feature"	Scores	Movie Features	?
"Score"	Total Score	**Score**	?

Application 1: Weight, Feature and Score

Meta-application: evaluation.
(Useful for ranking, comparing, etc.)

Goal: To evaluate a city, a university, an employee, a basketball player, etc.

Step 1: **Set up "rule"; specific to the evaluator**

Step 1.1 Set up "features" (indicators)

Step 1.2 Provide **weights** for different features.

Step 2: **Scoring each feature, for each object.**

Provide a "rating" for each feature, obtaining a feature vector

Step 3: **Compute "evaluation score"**

by computing the inner product of weight vector and feature vector

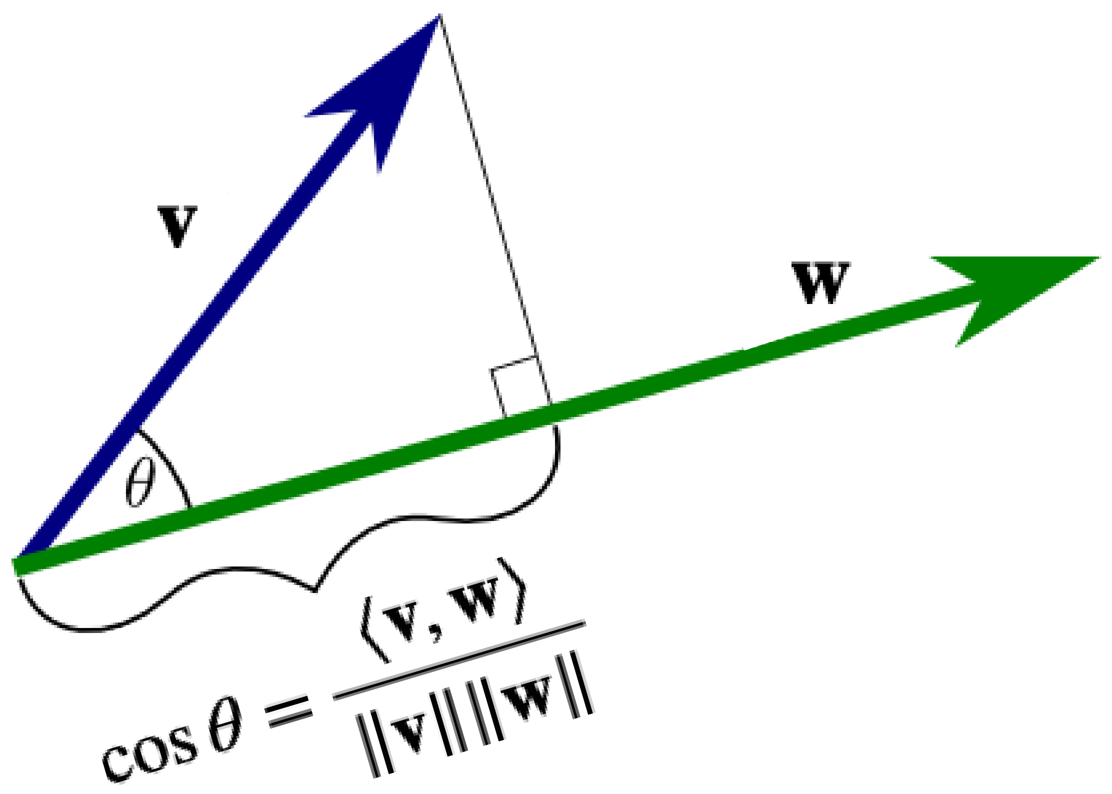
Remark: "Score" can be used in other areas, e.g. machine learning

Part III Inner Product and Norm

Cosine Similarity

Euclidean dot product formula

Cosine Similarity $\longrightarrow \cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|}$



Range: [-1,1].   The higher, the more similar.

Application: Searching

Searching: Given a query, find the most 10 relevant entities, e.g. sentences; websites

Idea: Vectorize + compute cosine similarity

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Searching: Given a query, find the most 10 relevant entities, e.g. sentences; websites

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Query Sentence: "How to make coffee?" ⟶ (0.8, 1.1, 0.5) = **q**

Candidate 1: "Procedure for preparing Latte" ⟶ (2.5, 4.3, 3.7) = **u**

Candidate 2: "Story of Car Maker" ⟶ (-1.5, 2.5, -0.3) = **v**

Application: Searching

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Cosine similarity of q and u is ~ 0.95;

Cosine similarity of q and v is ~ 0.33.

Compare cosine similarity: 0.95 > 0.33.

so candidate 1 is more similar to the query.

Calculation of Cosine Similarity

Just list the procedure for the first pair here

First Pair: $(0.8, 1.1, 0.5)$ **and** $(2.5, 4.3, 3.7)$

1. **Dot Product:**

$$\mathbf{A} \cdot \mathbf{B} = 0.8 \times 2.5 + 1.1 \times 4.3 + 0.5 \times 3.7 = 2 + 4.73 + 1.85 = 8.58$$

1. **Euclidean Norms:**

$$\|\mathbf{A}\| = \sqrt{0.8^2 + 1.1^2 + 0.5^2} = \sqrt{0.64 + 1.21 + 0.25} = \sqrt{2.1} \approx 1.4491$$

$$\|\mathbf{B}\| = \sqrt{2.5^2 + 4.3^2 + 3.7^2} = \sqrt{6.25 + 18.49 + 13.69} = \sqrt{38.43} \approx 6.2006$$

1. **Cosine Similarity:**

$$\text{Cosine Similarity} = \frac{8.58}{1.4491 \times 6.2006} \approx \frac{8.58}{8.9848} \approx 0.9549$$

Property 1: Relation of Dot Product and Norm

Cauchy–Schwarz Inequality

$$|\langle \mathbf{v} \cdot \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

An “incorrect” explanation:

$$\cos\theta = \frac{\langle \mathbf{v} \cdot \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} \leq 1$$

We will formally prove Cauchy–Schwarz inequality in approx. Week 8!

Property 2: Triangle Inequality

Triangle inequality

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Proof:

Property 3: Pythagoras Law (毕达哥拉斯定理)

Pythagoras Law

$$\|v\|^2 + \|w\|^2 = \|v - w\|^2 \quad \text{iff} \quad \langle \mathbf{v}, \mathbf{w} \rangle = 0$$

Proof:

Summary Today

Today, we have learned: $\|\mathbf{v}\| = \|\mathbf{v}\|_2 := \left(v_1^2 + \cdots + v_n^2\right)^{\frac{1}{2}}$

— Norm of vector

ℓ_2 norm a.k.a. (also known as) Euclidean norm

— Inner product of two vectors

— Applications of inner product:
"feature" and "score" for evaluation

--Properties:

Cosine similarity and Cauchy-Schwartz inequality
Triangular inequality

More Examples

Question: What if we have three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$

Can we write $\mathbf{x}$ as a linear combination of $\mathbf{y}, \mathbf{z}$?

Why do we need to do this in our real-world?

The next lecture!