Lecture 04

Solving Linear System I: "Good Case" of Square System

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Recall

In the last lecture …

- Definitions of linear equations and systems of linear equations
- Examples of solving 2×2 system of linear equations
- Definition of Matrix-vector product
- Definition of an augmented matrix representation

Today's Lecture

Today … More on System of **Linear Equations!**

After this lecture, you should be able to

- Tell the definition of lower and upper triangular matrices
- Tell what are elementary row operations, and why they are allowable
- Solve a linear system (square system) using Gaussian Elimination

Part I **Gauss-Jordan Elimination: 2 By 2 Example**

Length: 20-25 mins.

Recall: Augmented Matrix

Definition (Augmented Matrix)

Given a linear system,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the corresponding augmented matrix is:

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Overdetermined, Underdetermined and Square

Definition (System of Linear Equations)

An $m \times n$ system of linear equations is

(1) **overdetermined system** if $m > n$

(2) **underdetermined** system if $m < n$

(3) **square system** if $m = n$

Today we solve an $n \times n$ System (square)

An $m \times n$ matrix is

(1) tall, if

(2) wide, if

(3) square, if

Special Matrices

Definition (Lower Triangular Matrix)

A square matrix of the form

$$L = \begin{bmatrix} \ell_{1,1} & & & & 0 \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

is called a **lower triangular matrix.**

Mathematical definition: $L_{ij} = 0$, for any $1 \leq i < j \leq n$.

Special Matrices

Definition (Upper Triangular Matrix)

A square matrix of the form

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

is called an **upper triangular matrix**

Mathematical definition: $U_{ij} = 0$, for any $1 \leq j < i \leq n$.

Special Matrices

Definition (Diagonal Entry)

For a square matrix A each entry $A_{i,i}$ is called a diagonal entry of

Definition (Diagonal Matrix)

A square matrix D satisfying $D_{ij} = 0, \quad \forall i \neq j$

is called a diagonal **matrix.**

Examples

Are these matrices:
upper triangular matrices, lower triangular matrices, diagonal matrices?

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination for 2*2 System: Matrix View

Equation view

$$\begin{cases} x_1 + x_2 = 12 \\ 2x_1 + 4x_2 = 38. \end{cases}$$

Augmented Matrix view

Part II **Elementary Row Operations**

Length: 15-20 mins.

Operation on Rows?

Last page: Operation on equations; translate equations to matrices.

Can we translate operations (from equation-operations to matrix-operations)?

Equation view

$$\begin{cases} x_1 + x_2 = 12 \\ 2x_1 + 4x_2 = 38. \end{cases}$$

② − 2 × ①

$$(2x_1 + 4x_2 = 38)$$
$$- 2\,(x_1 + x_2 = 12)$$

$$2x_1 - 2x_1 \qquad 4x_2 - 2 \cdot x_2 \qquad 38 - 2 \cdot 12$$
$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

Throw away x_i's →

Augmented Matrix view

$$\begin{bmatrix} 1 & 1 & | & 12 \\ 2 & 4 & | & 38 \end{bmatrix}$$

Operation on Rows?

What row operation?

Last page:

$$\begin{bmatrix} 1 & 1 & | & 12 \\ 2 & 4 & | & 38 \end{bmatrix}$$

Perform operation:

$$\begin{bmatrix} 2 & 4 & | & 38 \end{bmatrix}$$
$$-2 \times \begin{bmatrix} 1 & 1 & | & 12 \end{bmatrix}$$

Get:

Express in row operation?

$$\begin{bmatrix} 1 & 1 & | & 12 \\ 2 & 4 & | & 38 \end{bmatrix}$$

Indicate the operation:

Another Row Operation

Equation view

$$x_1 + x_2 = 12 \quad ①$$
$$2x_2 = 14 \quad ③$$

Divide ③ by 2:

$$x_2 = \frac{14}{2} = 7 \quad ④$$

Update the linear system:

$$x_1 + x_2 = 12 \quad ①$$
$$x_2 = 7 \quad ④$$

Augmented Matrix view

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right]$$

Review

Solving a 2×2 system:

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 7 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \end{array}\right]$$

What are the key steps?

(1) Multiply a row by a **non-zero** scalar

(2) Add to one row a scalar multiple of another

(3) Swap the positions of two rows

Third Row Operation

Equation view

Augmented Matrix view

$$2x_1 + 4x_2 = 38 \quad ①$$
$$x_1 + x_2 = 12 \quad ②$$

→

$$\left[\begin{array}{cc|c} 2 & 4 & 38 \\ 1 & 1 & 12 \end{array}\right]$$

Swap the two equations first. →

$$x_1 + x_2 = 12 \quad ②$$
$$2x_1 + 4x_2 = 38 \quad ①$$

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right]$$

Remark:

You could eliminate ____ from Eq. 2, without swapping.

In this example:

You can swap them, but not absolute necessary.

Allowable Operations on Equations

(1) [**Multiplication**] Multiply an equation by a **non-zero** scalar

$$2x_1 + 4x_2 = 38 \quad \rightarrow$$

(2) [**Addition**] Add to one equation a scalar multiple of another

$$\begin{cases} x_1 + x_2 = 12, \\ 2x_1 + 4x_2 = 38 \end{cases} \quad \rightarrow$$

(3) [**Interchange**] Swap two equations

$$\begin{cases} x_1 + x_2 = 12, \\ 2x_1 + 4x_2 = 38 \end{cases} \quad \rightarrow$$

Operations on linear equations!

Allowable Operations on Rows

Definition (Elementary Row Operations)　　(初等行变换)

(1) [**Multiplication**] Multiply a row by a **non-zero** scalar

$$R_i \to$$

(2) [**Addition**] Add to one row a scalar multiple of another

$$R_i \to$$

(3) [**Interchange**] Swap the positions of two rows

$$\begin{pmatrix} R_i \\ R_j \end{pmatrix} \to$$

Elementary Row Operations Preserves Solution

Exercise (The operations preserve solutions)

Performing elementary operations will create a new system.

Prove: The new system and the original system has the same solution(s).

Claim: $(S1)$ and $(S2)$ have the same solutions.

$$(\text{S1}) \quad \begin{cases} A_1 x = b_1 & ① \\ A_2 x = b_2 & ② \end{cases} \qquad\qquad (\text{S2}) \quad \begin{cases} A_1 x = b_1 & ① \\ \alpha A_2 x = \alpha b_2 & ③ \end{cases}$$

2nd Elementary Row Operations Preserves Solution

Claim (2nd row operation preserve solutions)

System (S1):

$$\text{(S1)} \quad \begin{cases} A_1 x = b_1 & ① \\ A_2 x = b_2 & ② \end{cases}$$

where $A_1, A_2 \in \mathbb{R}^{1\times 2}$, $x \in \mathbb{R}^2$, and $b_1, b_2 \in \mathbb{R}$.

System (S2):

After performing a row operation, the system transforms into:

$$\text{(S2)} \quad \begin{cases} A_1 x = b_1 & ① \\ (\alpha A_1 + A_2)x = \alpha b_1 + b_2 & ③ \end{cases}$$

$(S1)$ and $(S2)$ have the same solutions.

What Operation Does Not Work?

Judgement:

True or False:

Suppose the system of equations is given by:

$$(\text{P1}) \quad \alpha_1 x_1 + \alpha_2 x_2 = b_1,$$

$$(\text{P2}) \quad \beta_1 x_1 + \beta_2 x_2 = b_2.$$

After performing the row operation:

$$(\text{P1'}) \quad \alpha_1 x_1 + \alpha_2 x_2 = b_1,$$

$$(\text{P2'}) \quad (\alpha_1 \beta_1) x_1 + (\alpha_2 \beta_2) x_2 = \alpha_1 b_2,$$

we claim that $(P1)$ and $(P2)$ have the same solutions as $(P1')$ and $(P2')$.

Other Operations

Exercise (Other Operations)

Can the following operations be performed?

(4) Multiply a row by zero

(5) Multiply the coefficients of two equations

Part III **G-J Elimination Using Row Operations**

Length: 15-20 mins.

Gaussian Elimination: 3 by 3 System

Step 1: Forward Elimination (Equation)

$$\begin{aligned} x + \ \ y + \ \ z &= 6 \\ x + 2y + 2z &= 9 \\ x + 2y + 3z &= 10 \end{aligned}$$

$$\begin{aligned} x + y + z &= 6 \\ y + z &= 3 \\ z &= 1 \end{aligned}$$

Step 1: Forward Elimination (Matrix)

Gaussian Elimination

Step 2: Backward Substitution (Scalar)

$$
\begin{aligned}
x + y + z &= 6 \\
y + z &= 3 \\
z &= 1
\end{aligned}
$$

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}
$$

Step 2: Backward Substitution (Matrix)

First Step

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & | & b_n \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ 0 & \hat{a}_{22} & \cdots & \hat{a}_{2n} & | & \hat{b}_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & \hat{a}_{n2} & \cdots & \hat{a}_{nn} & | & \hat{b}_n \end{bmatrix}$$

Identify the leading entry in the first row, a_{11}

Use a_{11} as the pivot to eliminate all entries below it in the first column.

Perform row operations:

$$R_i \to R_i - \frac{a_{i1}}{a_{11}} R_1 \quad \text{for } i = 2, 3, \ldots, m.$$

First Step

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & | & b_n \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ 0 & \hat{a}_{22} & \cdots & \hat{a}_{2n} & | & \hat{b}_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & \hat{a}_{n2} & \cdots & \hat{a}_{nn} & | & \hat{b}_n \end{bmatrix}$$

Identify the leading entry in the first row, a_{11} and ensure it is non-zero.

Use a_{11} as the pivot to eliminate all entries below it in the first column.

Perform row operations:

$$R_i \rightarrow R_i - \frac{a_{i1}}{a_{11}} R_1 \quad \text{for } i = 2, 3, \ldots, m.$$

Assumption for now (good case): a_{11} is nonzero.

Second Step

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ 0 & \hat{a}_{22} & \cdots & \hat{a}_{2n} & | & \hat{b}_2 \\ 0 & \hat{a}_{32} & \cdots & \hat{a}_{3n} & | & \hat{b}_3 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & \cdots & \hat{a}_{nn} & | & \hat{b}_n \end{bmatrix} \longrightarrow$$

1. **Identify the pivot**: The pivot for this step is $\hat{a}_{22}$, the updated entry in the second row and second column.
2. **Eliminate entries below the pivot**: Use $\hat{a}_{22}$ to eliminate all entries below it in the second column:

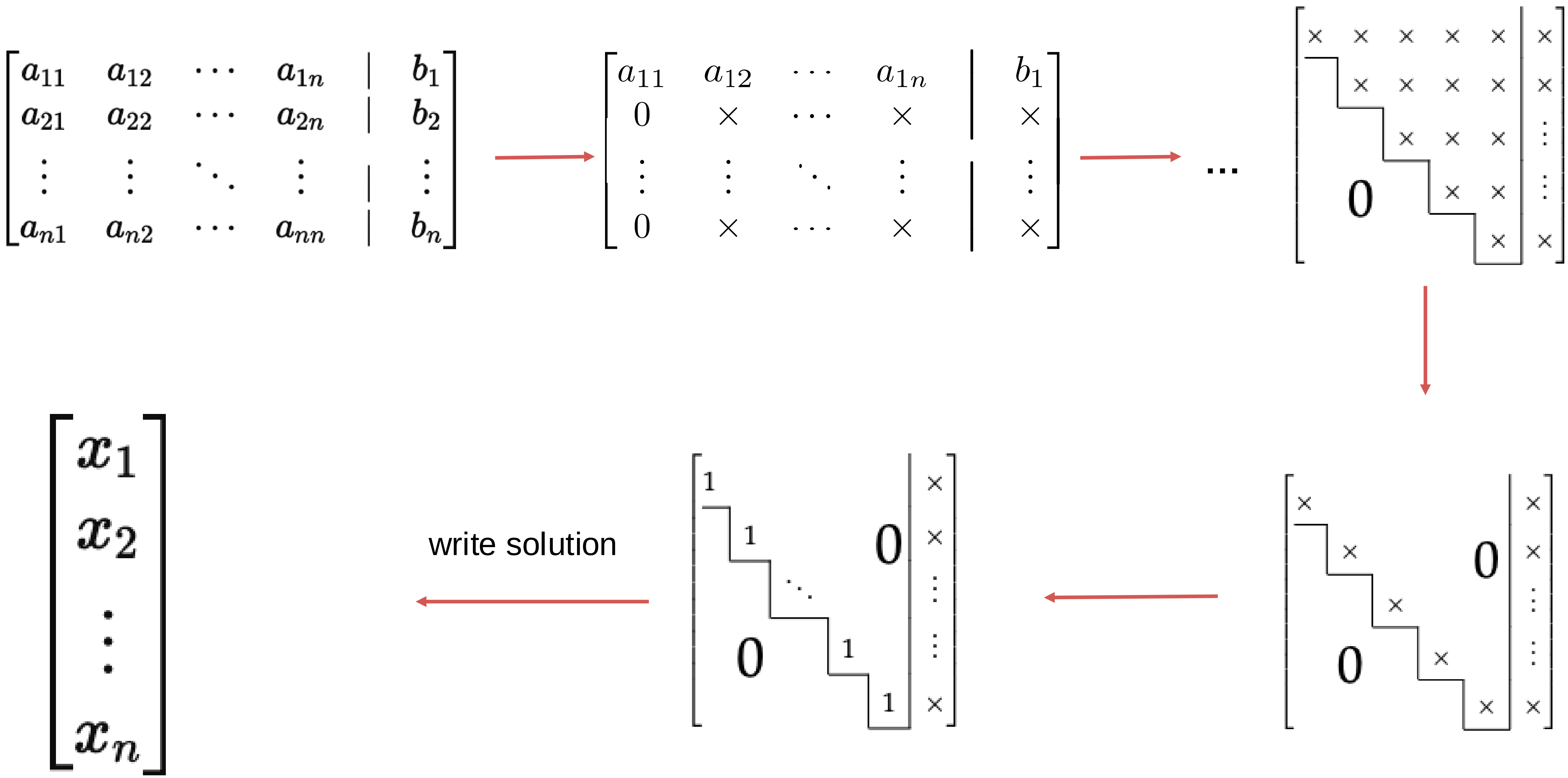
$$R_i \to R_i - \frac{\hat{a}_{i2}}{\hat{a}_{22}} R_2 \quad \text{for } i = 3, 4, \ldots, m$$

3. **Update the matrix**: The rows below the second row are updated accordingly, resulting in a modified matrix.

Assumption for now (good case): $\hat{a}_{22}$ is nonzero.

Gaussian-Jordan Elimination for "Good" Systems

Pipeline



(Think: does it always work?)

Summary of GJ-E for Solving Square Systems

Phase 1: Forward Elimination.

Perform elementary row operations and try to get an upper triangular matrix.

Phase 2: Backward substitution

Perform elementary row operations and try to get a diagonal matrix.

Assumption 1
At each iteration of the forward elimination, the next diagonal entry is nonzero.

Claim 1 Under Assumption 1, we can get a diagonal matrix at the end of Step 2.

Corollary 1 Under Assumption 1, the system has a unique solution.

This assumption may not hold for some problems; will discuss later.

Concluding Section

Summary Today

One sentence summary:

Detailed summary:

Questions:

Can we use matrix operations to represent GE?