Lecture 01

Introduction to Linear Algebra and Data Science

香港中文大學(深圳) | 数据科学学院
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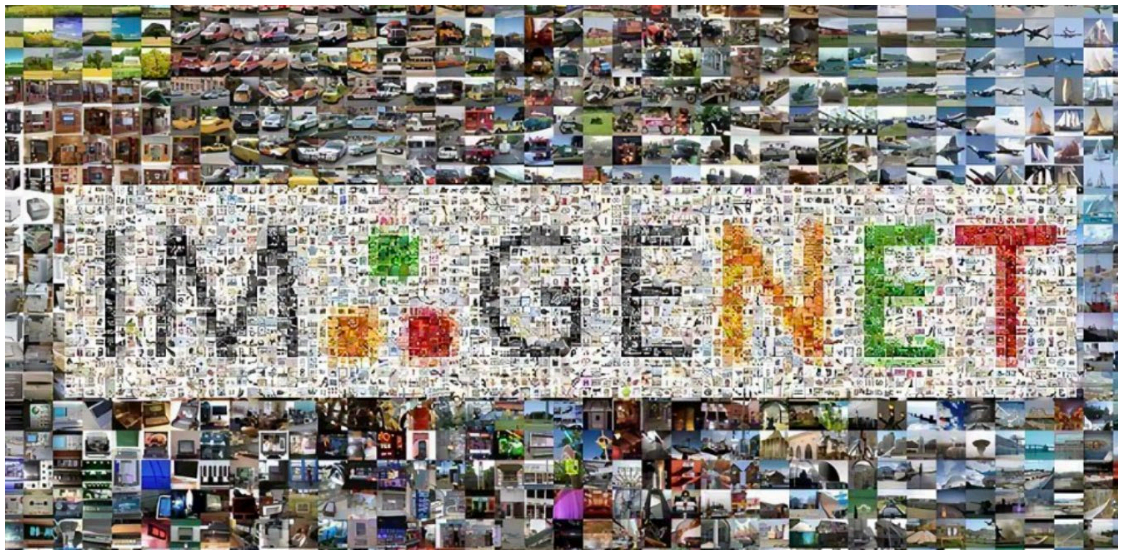
Why Data are Important

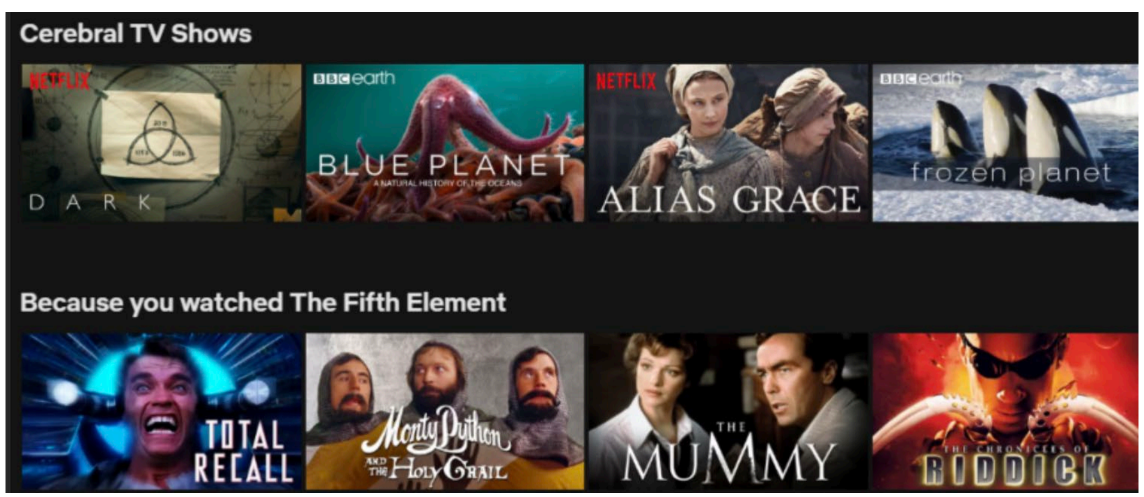
Data are the new "oil" and "power"!

Oil/power can drive cars, airplanes, cellphones, … — Hardware

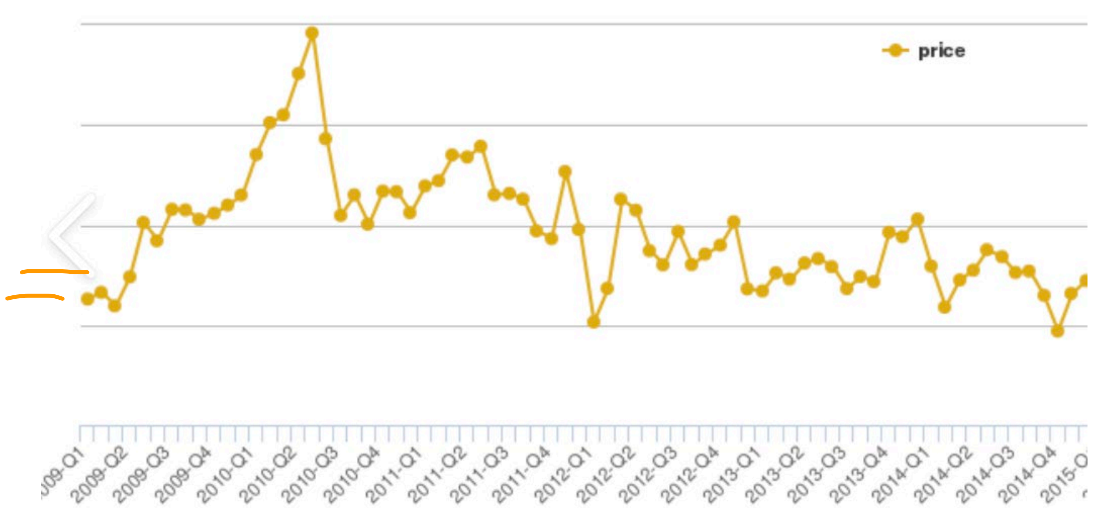
Data can drive AI and decision making, — "Software"

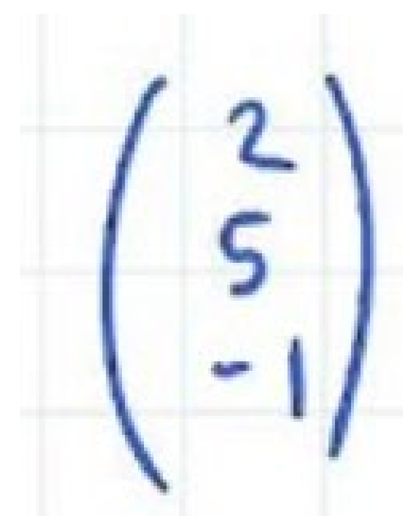
What are “Data”?





First Challenge: How to “Express” them?

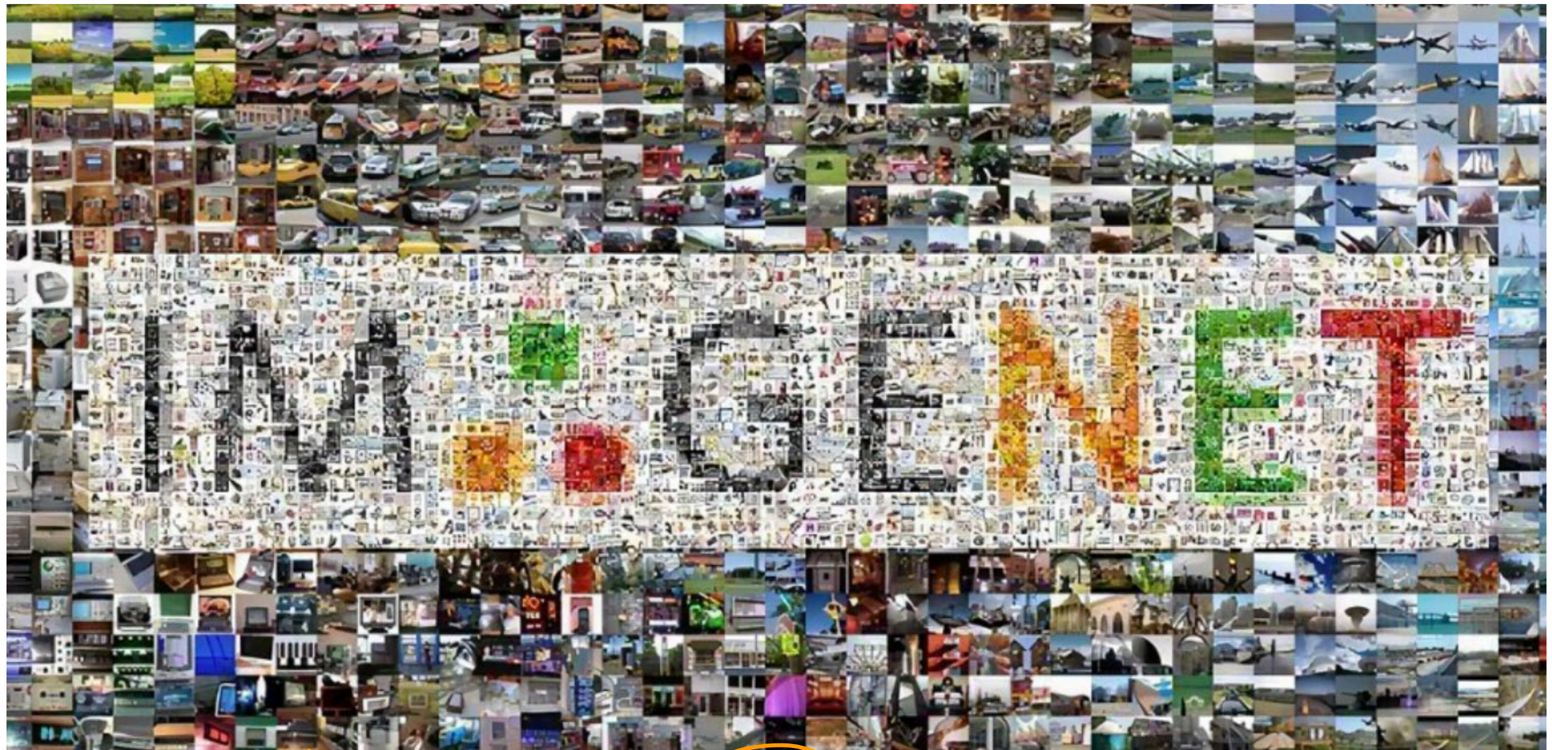




Figure?

vector!

First Challenge: How to "Express" them?



(200, 110, 30)

Figure?

157	153	174	168	150	152	129	151	172	161	155	156
155	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	106	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234

Matrix!

What is Linear Algebra?

Algebra: Mathematical representation of problems and operations

X, Y, x, y, A, B, …

addition +
subtraction -
multiplication ×
inner product ⟨·,·⟩ …

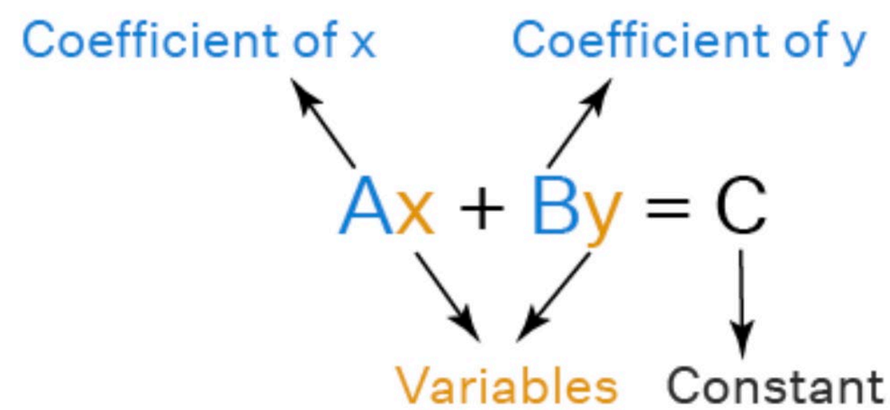
What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = 0$

Linear Algebra



Standard form of an Linear Equation

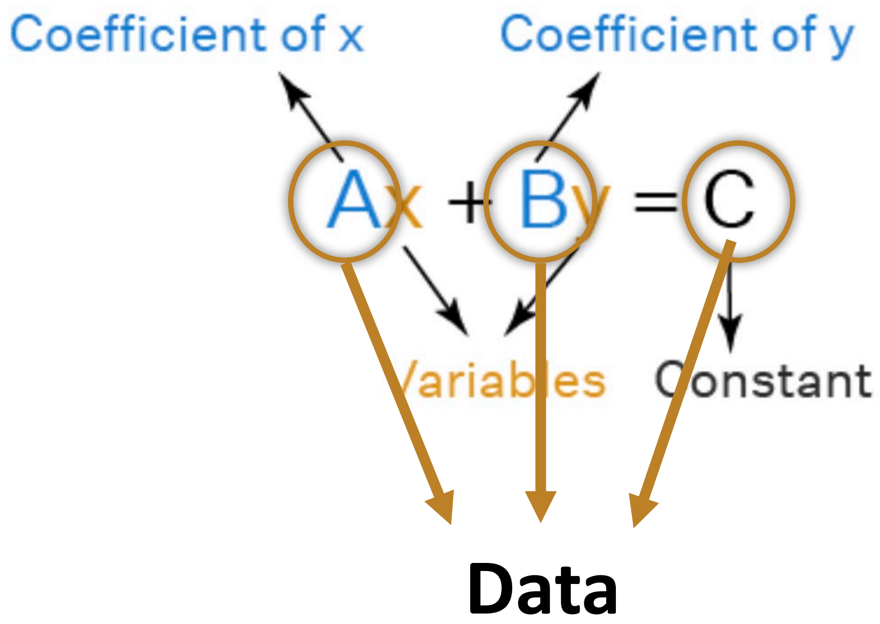
(no product between variables)

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra



Standard form of an Linear Equation

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra

$$A\mathbf{u} + B\mathbf{v} = \mathbf{c}$$

Matrices of **Data**

Standard form of a **System** of Linear Equations

What is Linear Algebra?

Primary/High-School Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Linear Algebra

$$A\mathbf{u} + B\mathbf{v} = \mathbf{c}$$

Vectors of Parameters/Data

Standard form of a **System** of Linear Equations

What is Linear Algebra?

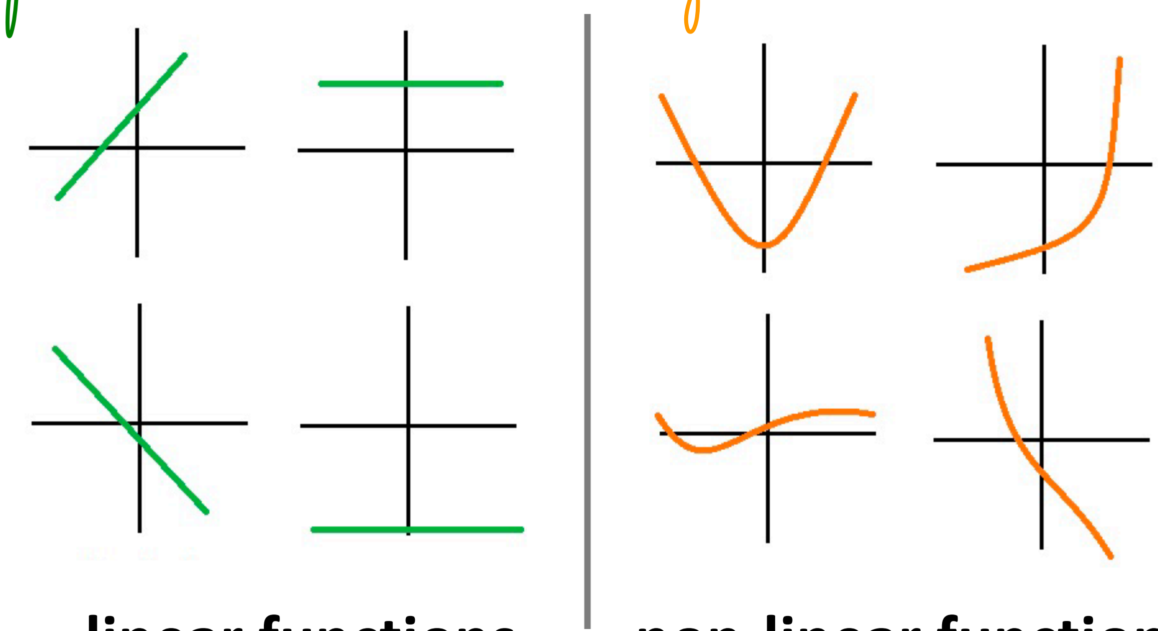
Algebra: Mathematical representation of problems and operations

X, Y, x, y, A, B, …

addition +
subtraction -
multiplication ×
inner product ⟨·,·⟩ …

Linearity:

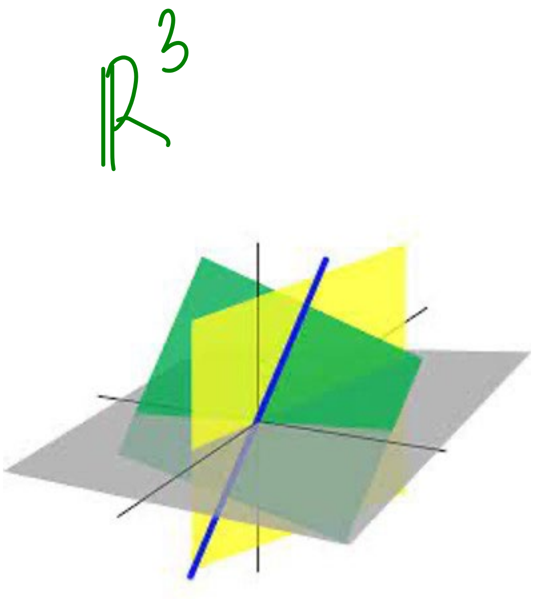
$y = ax + b$ $y = x^2 - 1$



linear functions **non-linear functions**

2-D representation

What is Linear Algebra?



Mathematically, **vector spaces**

and linear transformations

Applications

- **Machine learning and data science**
- Computer vision and graphics
- Graph theory
- Control theory
- Cryptography
- Fractals and chaos
- Energy systems
- Network systems
- Genetics
- Etc. …

Practically, it can be applied to **any** problems with

vector, matrix-type data, and linear models

Why Need Linear Algebra? A1: Fundamental

Quora — Search for questions, people, and topics

What exactly is linear algebra? Why do we need it?

Why study linear algebra?

Ask Q

Asked 9 years, 8 months ago Modified 3 years, 4 months ago Viewed 133k times

128

Simply as the title says. I've done some research, but still haven't arrived at an answer I am satisfied with. I know the answer varies in different fields, but in general, why would someone study linear algebra?

https://math.stackexchange.com/questions/256682/why-study-linear-algebra

Linear algebra is beyond important, it is fundamental to so many fields that I cannot count them all.

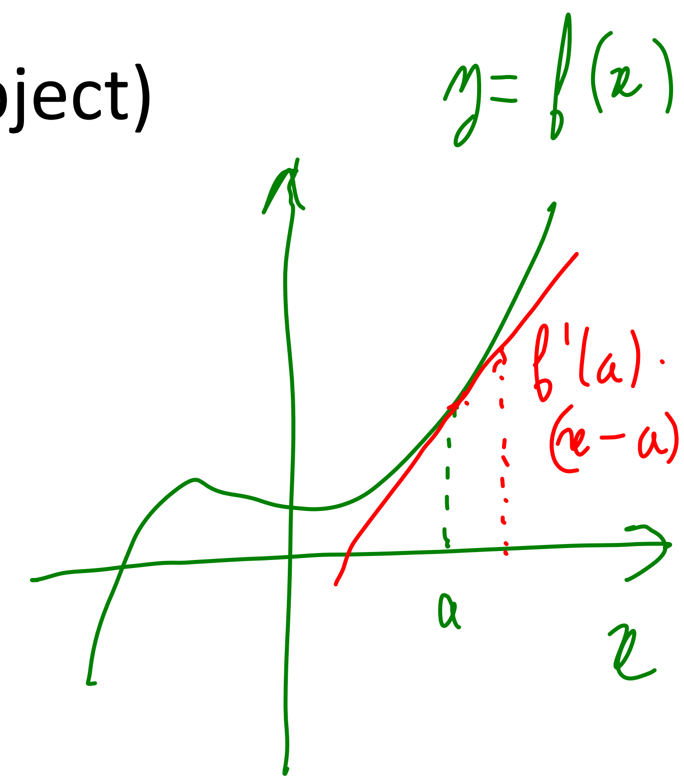
Having studied Engineering, I can tell you that Linear Algebra is fundamental and an extremely powerful tool in **every single** discipline of Engineering.

3D computer graphics? Linear algebra. Quantum mechanics? Linear algebra. Weather forecast models? Linear algebra.

Study it if you are into economics, computer graphics, physics, chemistry, statistics or anything quantitative (in today's world, that's everything).

Why Learning Linear Algebra is Critical?

- It can be applied to **many** problems
- **Easy** to model, analyze, and compute (not an easy subject)
- Foundations of more advanced and complex methods



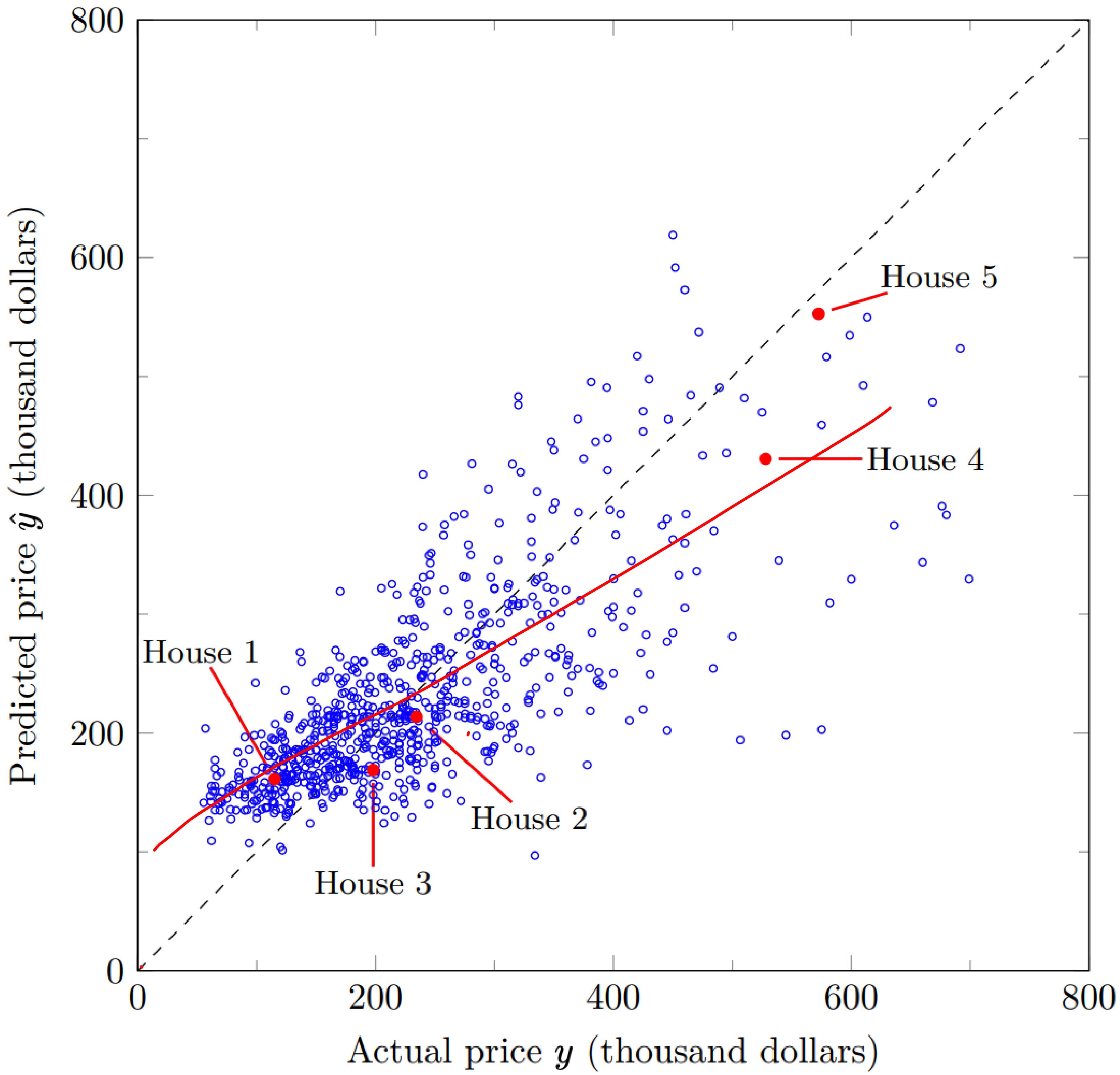
possibly non linear

Linear Models	Nonlinear Models
Less accurate	More accurate
Easy	Hard

$$f(x) = f'(a)(x-a) + \varepsilon \quad \text{(Taylor exp.)}$$

linear

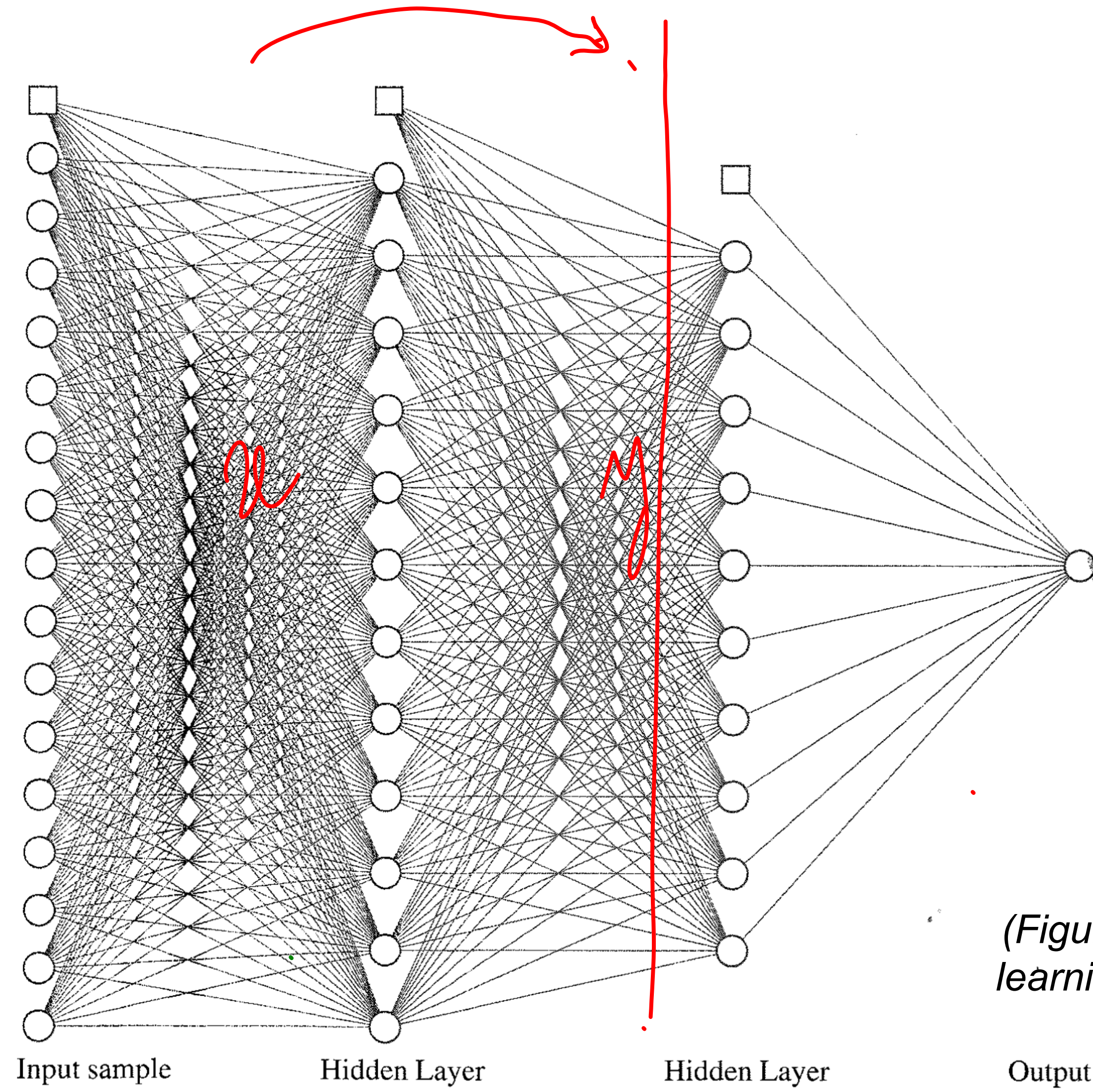
Why Learning Linear Algebra is Critical?



Actual and predicted sale prices of houses in Sacramento during 5 days

Example

Convolutional Neural Networks (CNNs)



nonlinear

$$y = \sigma(Ax)$$

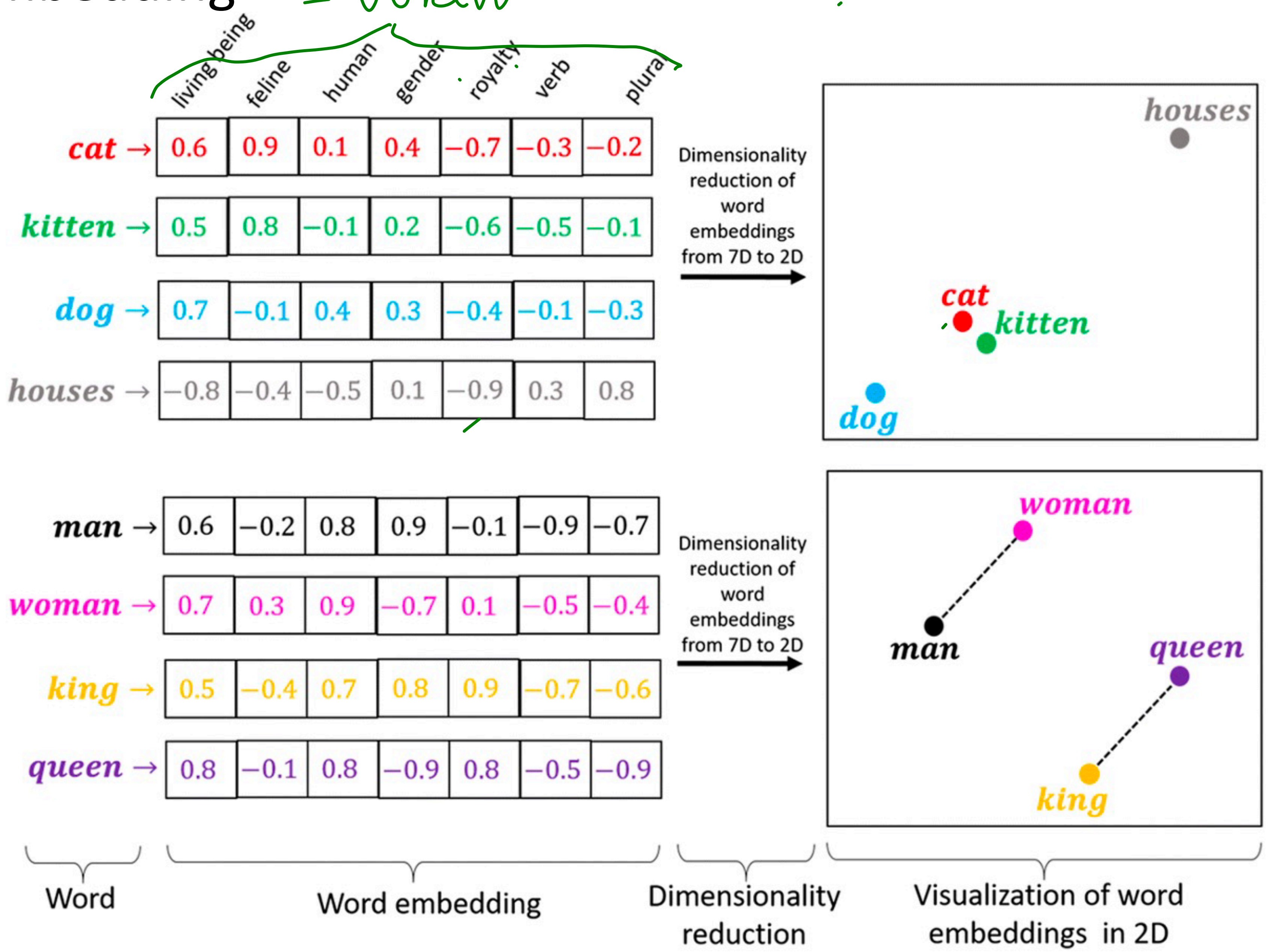
linear opera

(Figure resource: linear algebra and learning from data by Gilbert Strang)

- Each diagonal is a weight to be learned by optimization
- Edges from the square contain bias **vectors** and the other weights are shared in **matrices**

Example

Word Embedding



(Figure resource: https://medium.com/@hari4om/word-embedding-d816f643140)

This Course

Not a pure math course

For instance, we will not dive into the "axioms" when studying vector spaces

More data science problems will be demonstrated to motivate the concepts (compared with MAT2040)

For instance, least square problems, graph matrices, loss functions, regression model, convolution, etc. (will cover as much as possible)

Learning Methodology

Slides are inefficient for learning **math** …

You are strongly encouraged to take some notes

Mathematical definitions, lemmas, theorems, and formal expressions will be marked in a colored box. Writing them down will help memorize.

Examples, remarks, and other useless words will not be emphasized

General Goal

Classic linear algebra training:

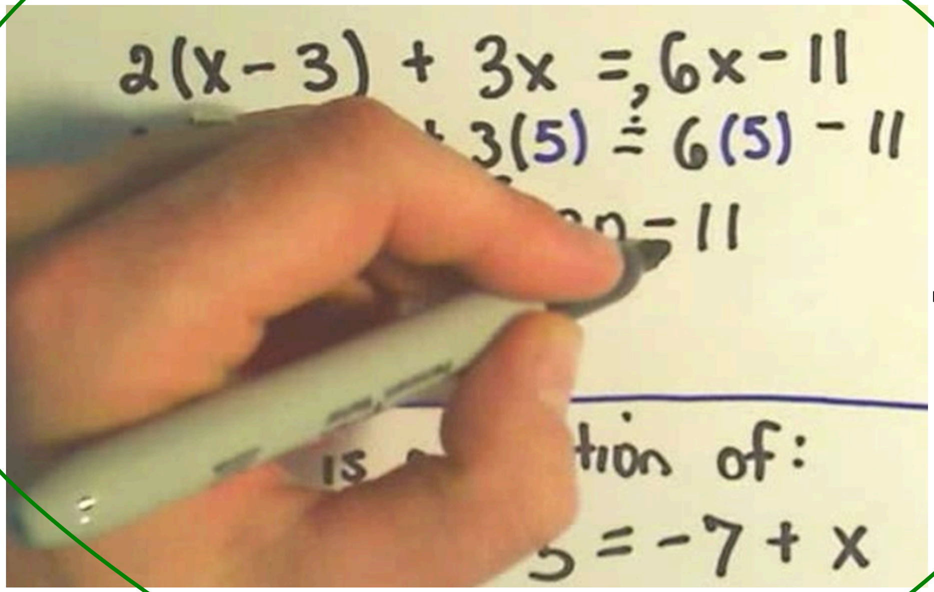
How to solve linear equations as fast as possible as humans?

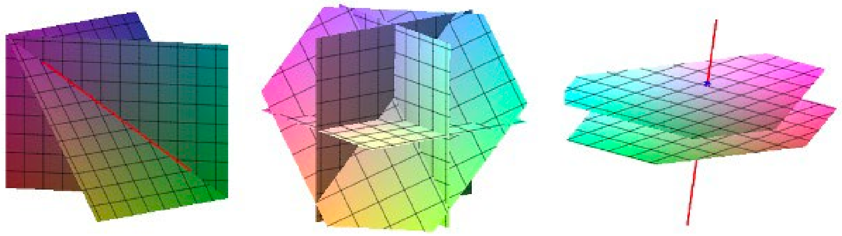
Computers will do the job!

(Unfortunately, in your exams, you sometime need to solve linear equations by hands; just occasionally!)

Learn the key ideas and the intuition/reasons behind!

General Goal





Learn the key ideas and the intuition/reasons behind!

From **How** to **Why (**keep asking yourself why**)**

Basic Components in Linear Algebra

Question1: What can you think about the basic components in Linear Algebra from your high school knowledge?

Vectors, matrices, and their operations etc

Question2: What are the most fundamental component in Linear Algebra?

Vectors!

Examples of Vectors

"The world is continuous, but the mind is discrete"

- David Mumford

How to interpret ?

Vectors

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]$$

(row) vector → $\mathbf{v}$

entry/element → v_i

Vectors

Definition (column vectors)

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^T$$

T → Transpose

$\mathbf{v}$ → column vector

v_i → entry/element

$$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \equiv (v_1, \ldots, v_n)$$

also written as

2x1 matrix = v

A $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ matrix

v $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

because we want to multiply vectors on the right of matrices

Convention: vectors considered as columns

Examples of Vectors

Example:

Zero Vector

$$\mathbf{v} = (0,0,0,\ldots,0) \in \mathbb{R}^p$$

p

$u \in \mathbb{R}^p$

$v + u = u$

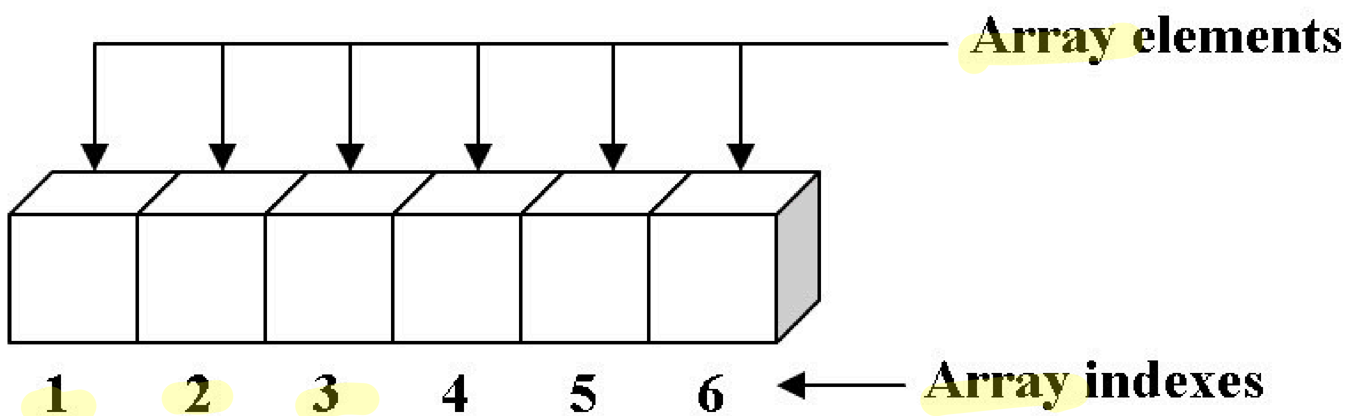
One Vector

$$\mathbf{v} = (1,1,1,\ldots,1)$$

Examples of Vectors

Example:

An **array** data structure in computer algorithms

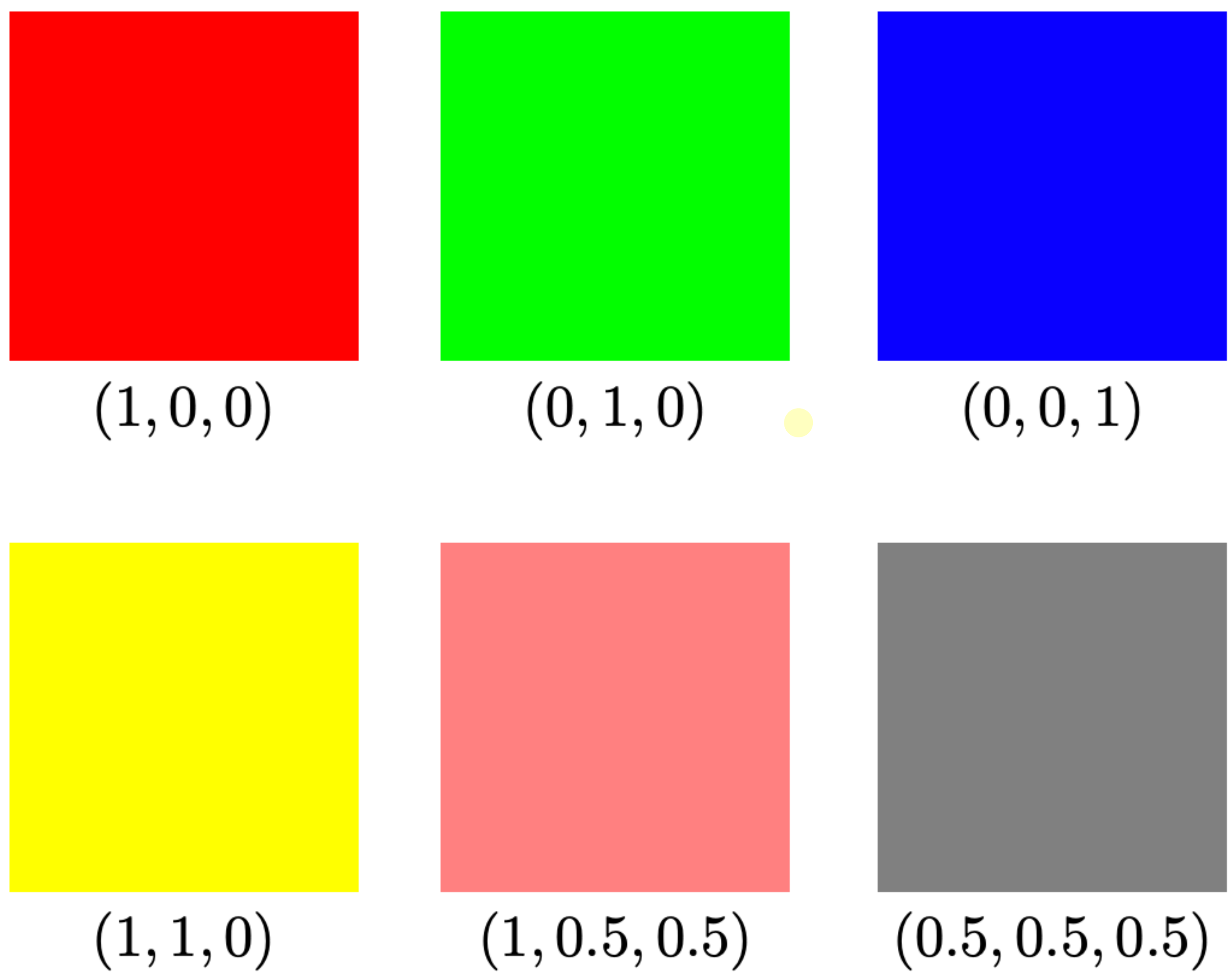


One-dimensional array with six elements

Examples of Vectors

Example:

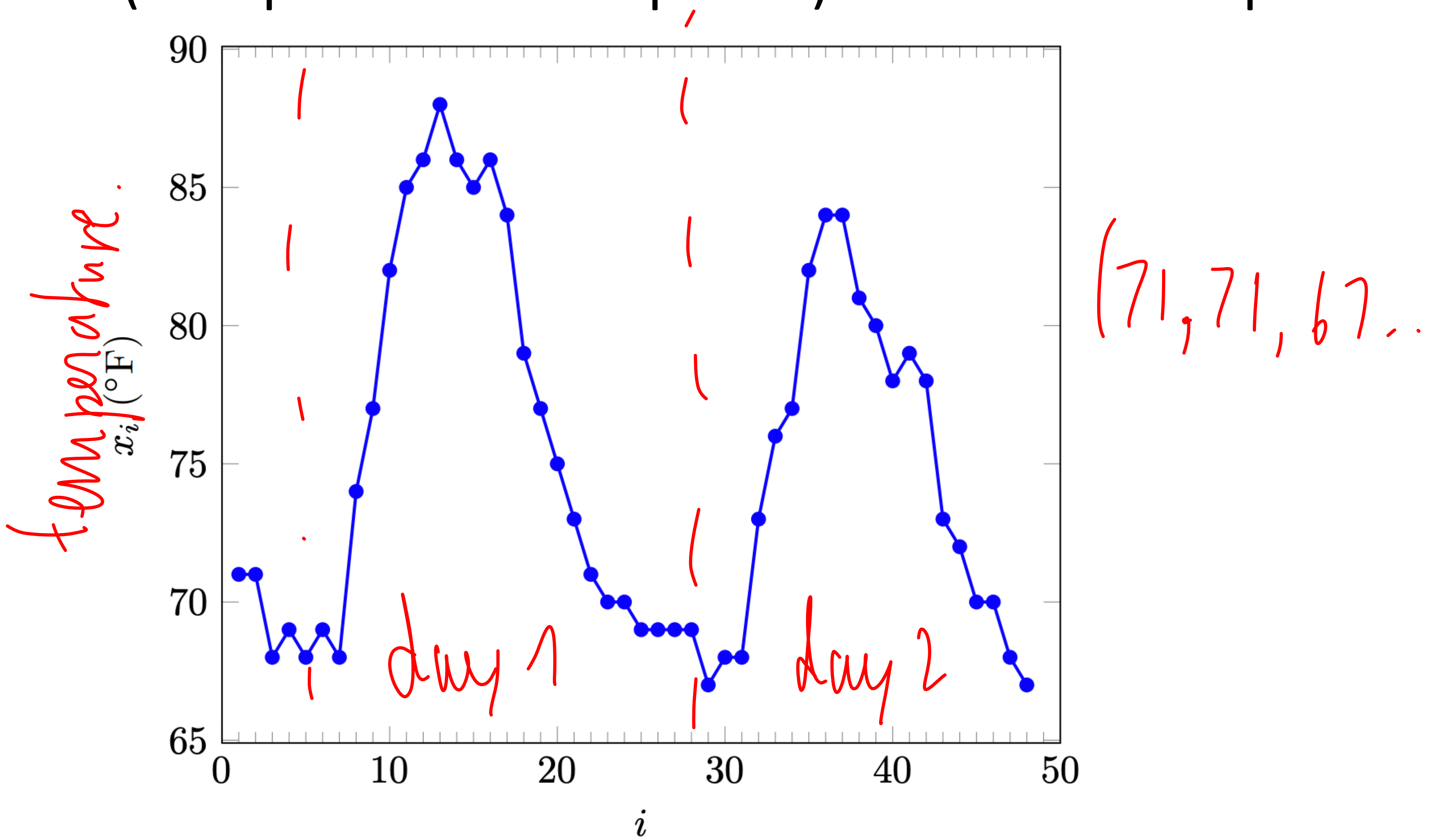
Color RGB Vectors



Examples of Vectors

Example:

A **time series** (a sequence of data points) in data science problems

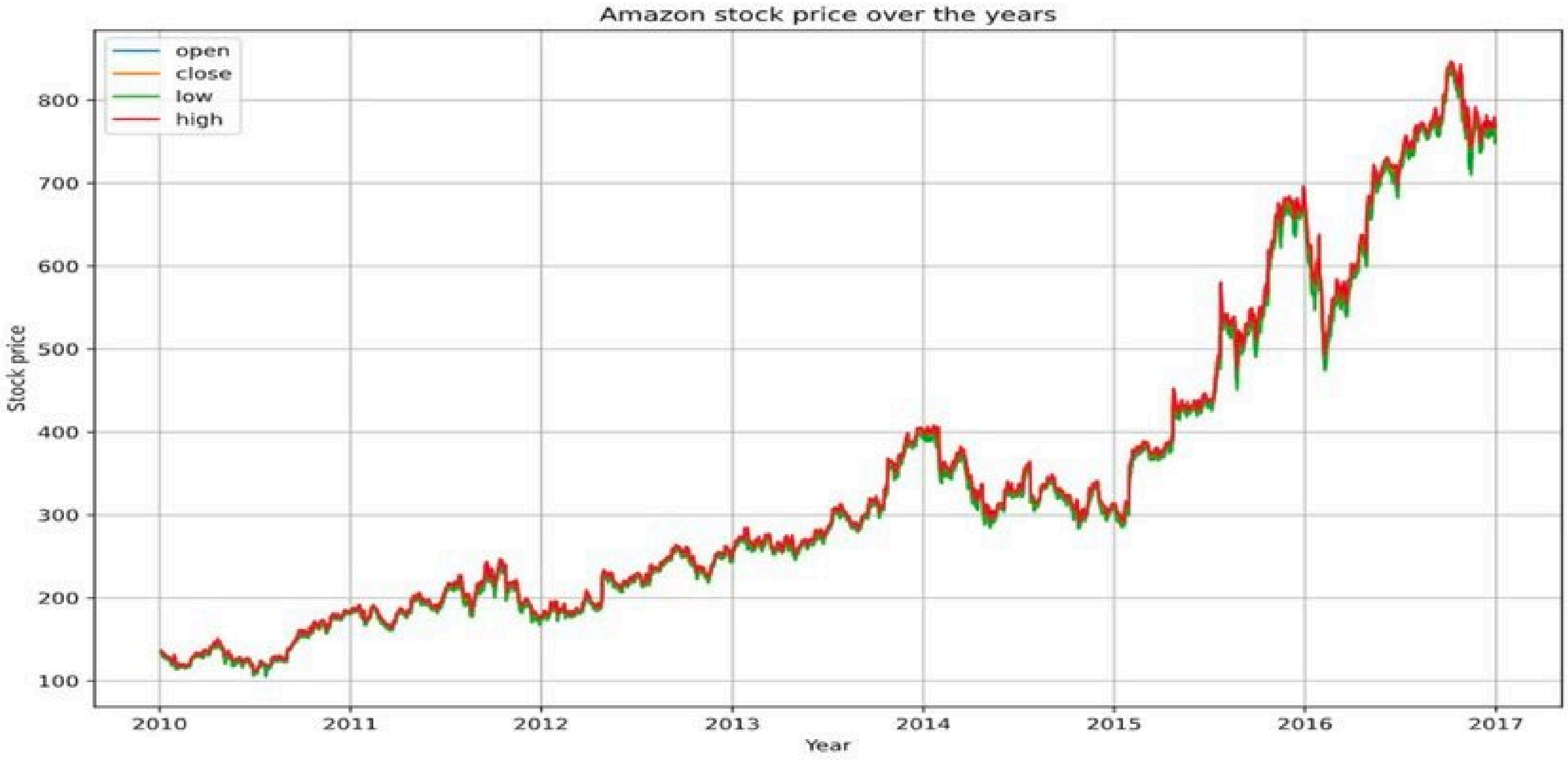


Hourly Temperature in LA on Aug 5 and 6, 2015

Examples of Vectors

Example:

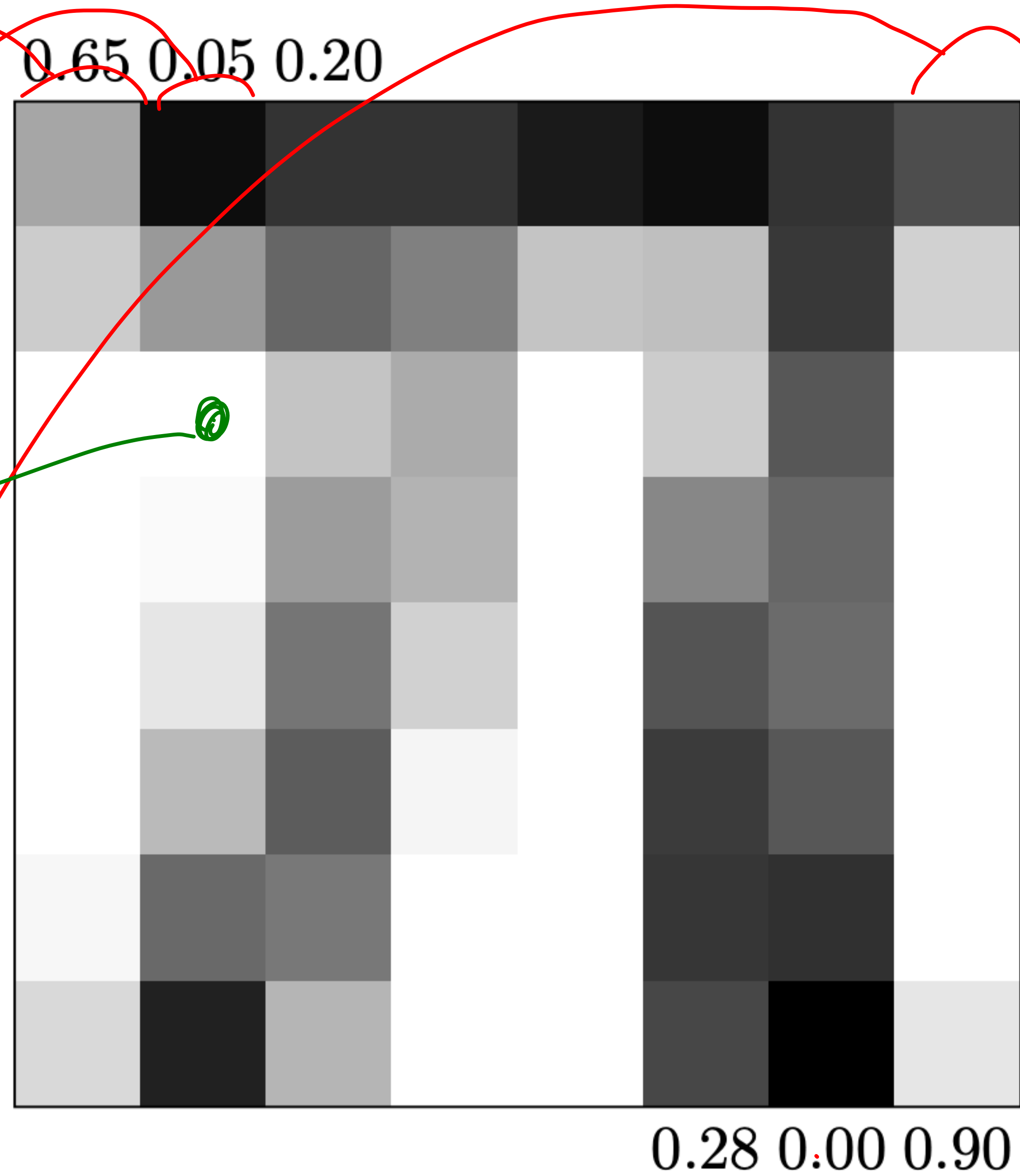
(In Finance) Stock Prices



Examples of Vectors

Example:

Images



Vectors

$$\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots, v_n]^{\mathsf{T}}$$

Transpose

column vector

entry/element

Example:

A **list** data structure in computer algorithms

A time series (a sequence of data points) in data science problems

Vector Operations

Vector Addition

$$\mathbf{v} = (v_1, \ldots, v_n)$$

$$\mathbf{w} = (w_1, \ldots, w_n)$$

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, \ldots, v_n + w_n)$$

Element-wise Operations! → same size

Vector Multiplication

multiplication with a scalar $c \in \mathbb{R}$

$$\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$$

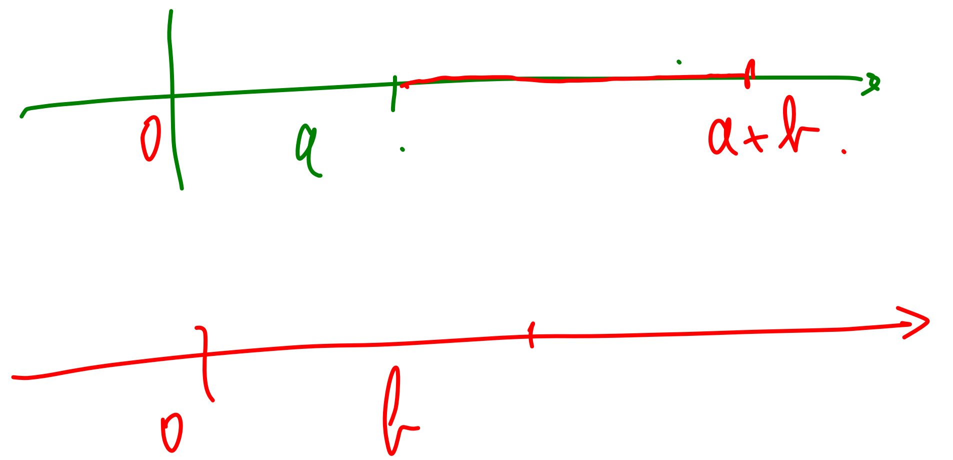
$$c\mathbf{v} = (cv_1, \ldots, cv_n)$$

Remark: *We do not worry about the domain of the elements* $v_1, \ldots, v_n$ *and* c *so far. They can be chosen from the set of real/complex numbers*

Visualization of Vector Operations

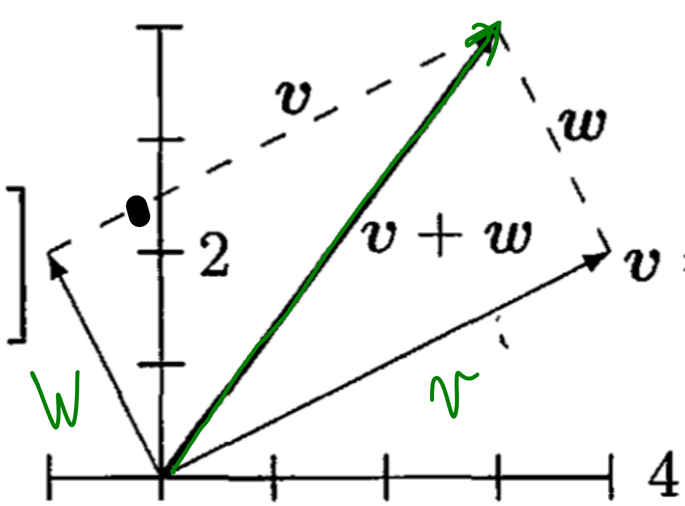
Addition $\quad w = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\in \mathbb{R}^2$

For reals a and b



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

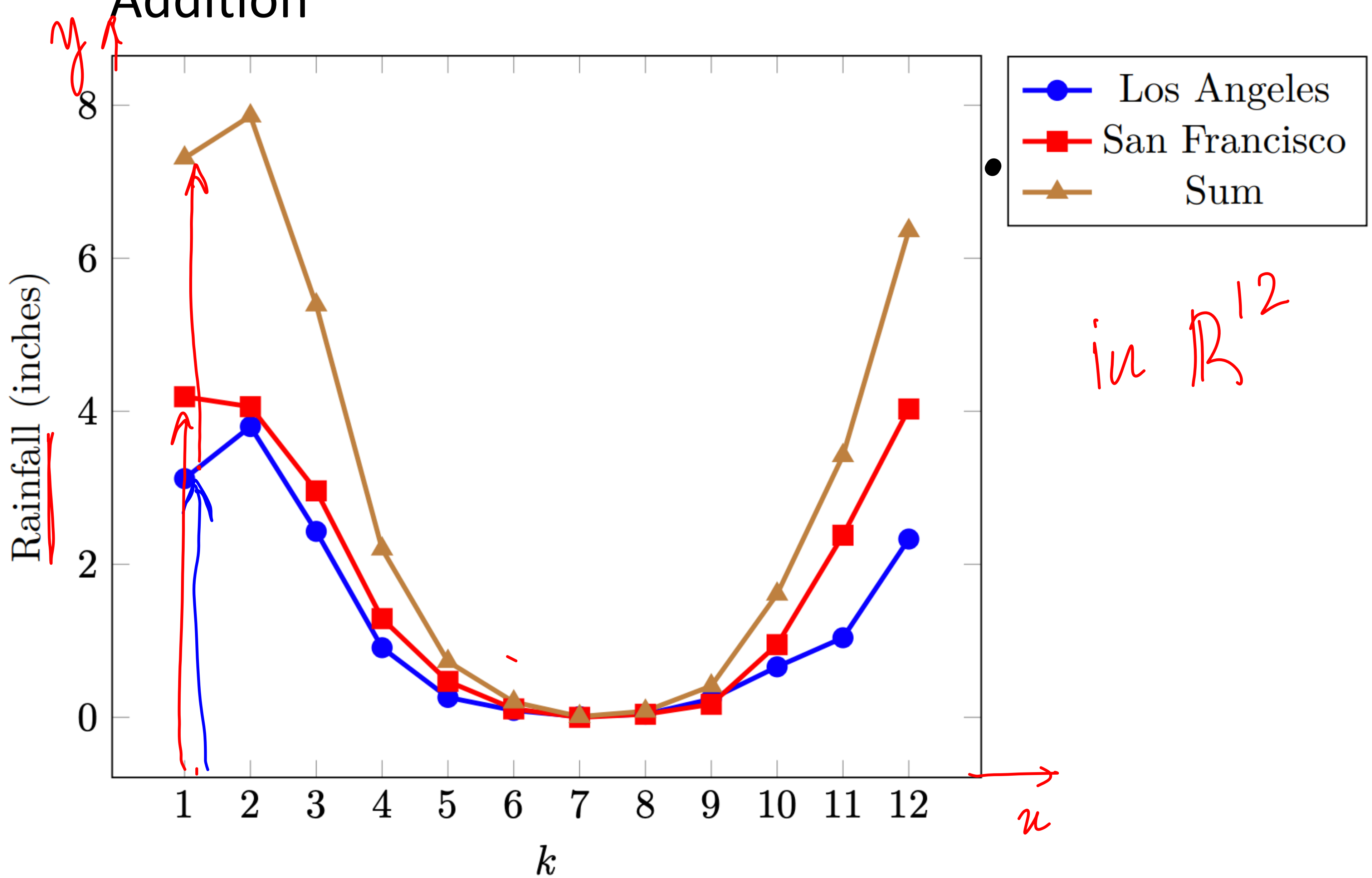
$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Example of Vector Addition

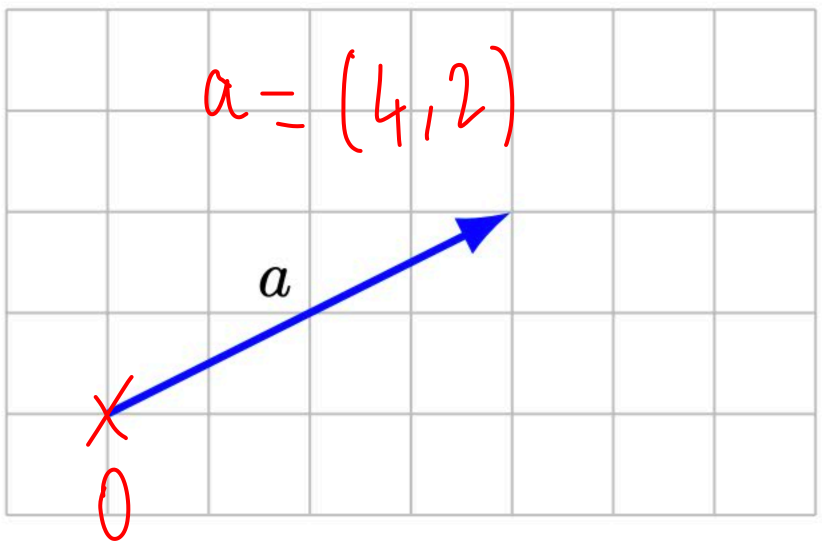
Addition

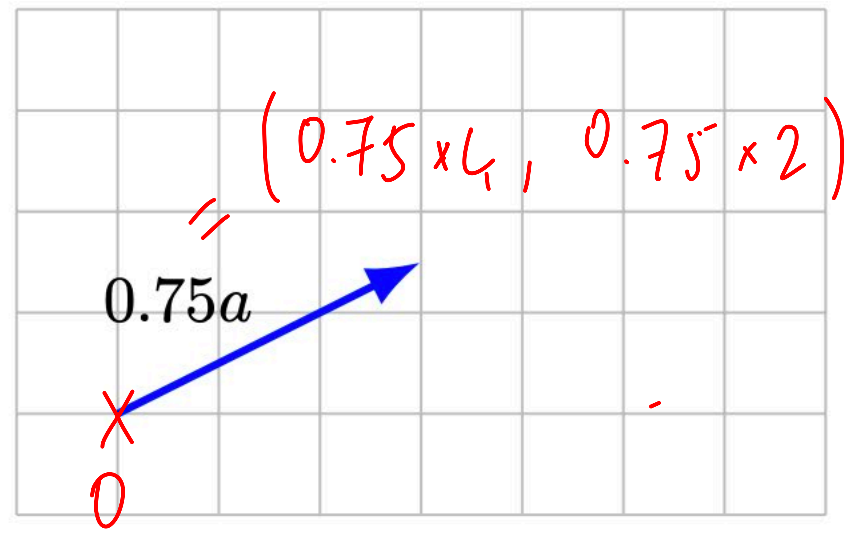


Average Monthly Rainfall in LA and SF

Visualization of Vector Operations

Multiplication (by a scalar)

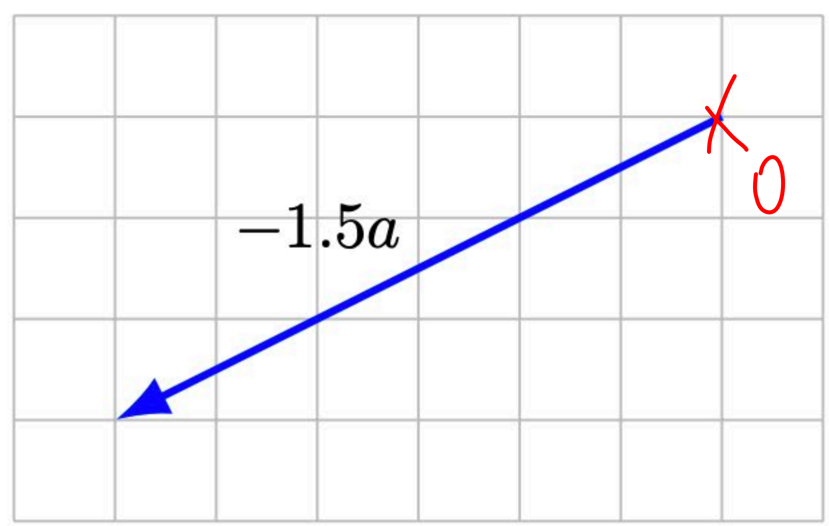




$c \in \mathbb{R}$

$v = (v_1, \ldots, v_n)$

$cv = (cv_1, \ldots, cv_n)$



Vector Operations

Linear Combination

$$\mathbf{v} = (v_1, \ldots, v_n) \quad \in \mathbb{R}^n$$

$$\mathbf{w} = (w_1, \ldots, w_n) \quad \in \mathbb{R}^n$$

$$c\mathbf{v} + d\mathbf{w} = (cv_1 + dw_1, \ldots, cv_n + dw_n)$$

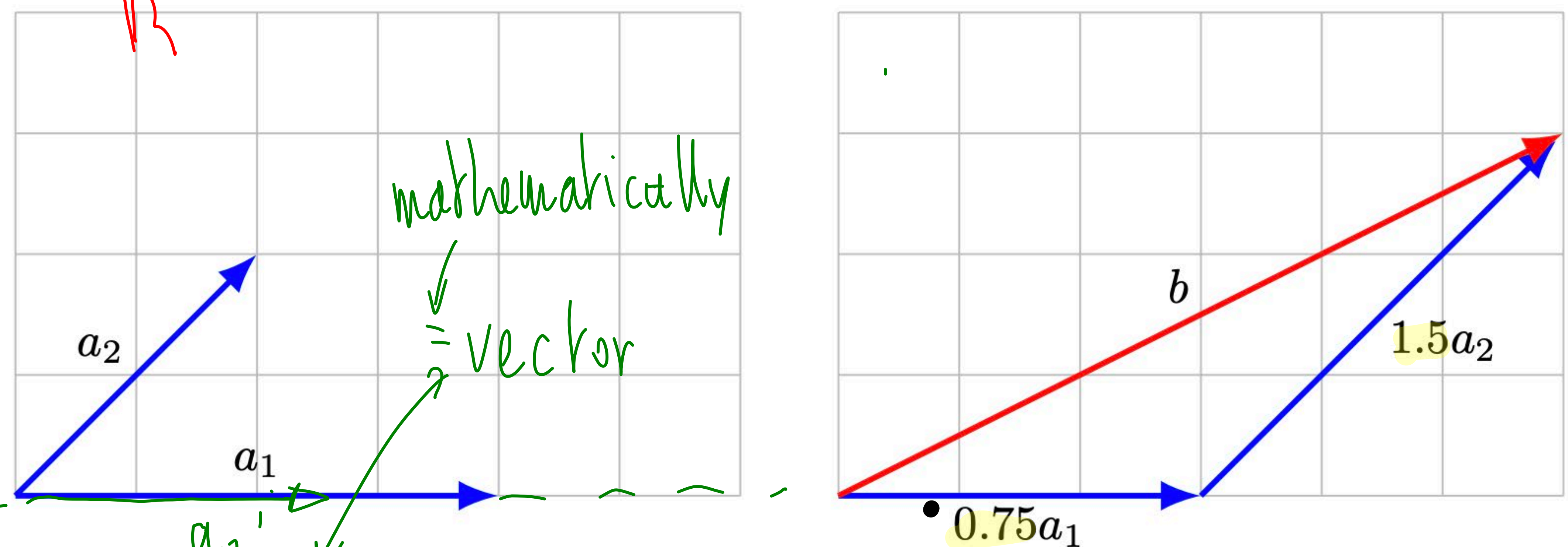
"$\in$" "belongs to" — $c \in \mathbb{R}$, $d \in \mathbb{R}$

We only need "**addition**" and "**multiplication**" to derive the "**linear combination**" law

Visualization of Linear Combination

Linear Combination

$\mathbb{R}^2$



any point of $\mathbb{R}^2$ is a linear combination of a_1 and a_2

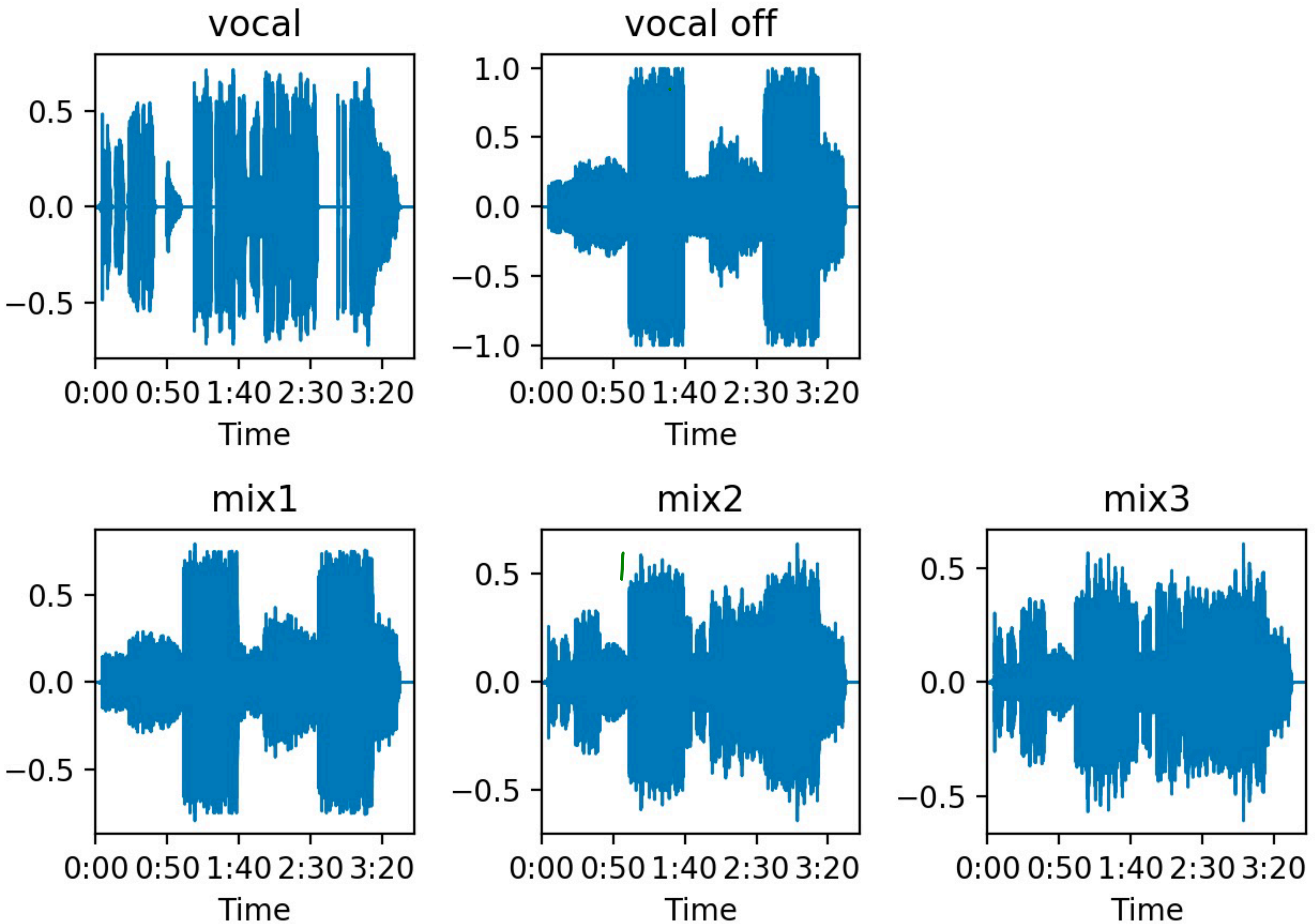
wrong if $a_2 = 0.75\, a_1$.

→ a_1 and a_2 need to be linearly independent

Examples of Linear Combination

Linear Combination

Example: Video/audio mixing



More Examples

Example (Combinations that are not linear)

$$\mathbf{u} = (u_1, u_2)$$

$$\mathbf{v} = (v_1, v_2)$$

product between entries.

$$f(\mathbf{u}, \mathbf{v}) = (u_1 v_1, u_2 + v_2)$$

Combination, but NOT linear combination!

More Examples

Example (Linear Combination of Vectors)

What combination $c\mathbf{v} + d\mathbf{w}$ produces $\mathbf{u}$?

$$\mathbf{v} = (1,2)$$

$$\mathbf{w} = (3,1)$$

$$\mathbf{u} = (14,8)$$

Answer:

[redacted]

find c and d such that $u = cv + dw$

More Examples

Example (Vector Operations)

Find $\mathbf{v}$ and $\mathbf{u}$ such that

$$\mathbf{u} + \mathbf{v} = (4,5,6)$$

$$\mathbf{u} - \mathbf{v} = (2,5,8)$$

Answer:

[redacted]

Summary

In today's lecture, we have covered

- Definition of **vectors**
- Vector Operations

(Textbook Section 1.1)

(Slides/Notes can be found in our course webpage)

Vector Operations

Question: Can you think about any other vector operations?

The next lecture!

A General Question: What if we have three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$

Can we write $\mathbf{x}$ as a linear combination of $\mathbf{y}, \mathbf{z}$?

The next next lecture!