Lecture 03

Systems of Linear Equations I: Forms and Elimination

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Recall

In the last lectures …

- Definition of norm and dot (inner product)
- Calculation of vector norms and inner products
- Real-world examples

Today

Today …   System of **Linear Equations!**

After this lecture, you should be able to

1. Write the 4 forms of **systems of linear equations**
2. Write various forms of **matrix-vector product**
3. Solve a linear system by **Gaussian elimination**

Reading Material: Logic and Proof

Difficulties in Linear Algebra, partially due to:

- lack of training of proving
- lack of LOGIC.

"Assume" "if" "then" "since" "if and only if"

logic is like telling a story

Material to check: if the story is not convincing it means it is not logic.

https://www.math.toronto.edu/preparing-for-calculus/ 3_logic/we_2_if_then.html

Part I System of Linear Equations

Linear System of Equations: Preliminary School Example

Problem (Chicken-Rabbit Problem 鸡兔同笼)

There are 35 heads and 94 feet in a cage.
How many chickens and how many rabbits are there?

Assumption: Each chicken has 1 head and 2 feet, and each rabbit has 1 head and 4 feet.

Introduce x = number of rabbits
y = number of chicken.

The pb expresses:

$$\begin{cases} 1y + 1x = 35 & (1) \\ 2y + 4x = 94 & (2) \end{cases}$$

$(2) - 2(1)$: $0 + 2x = 24$.

Linear Equations

Definition (Linear Equations)

A **linear equation** is the equation of the form

$$(1) \qquad a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

"or unknowns"

where $a_1, a_2, \ldots, a_n, b$ are real numbers and $x_1, x_2, \ldots, x_n$ are variables

Write it in the vector form:

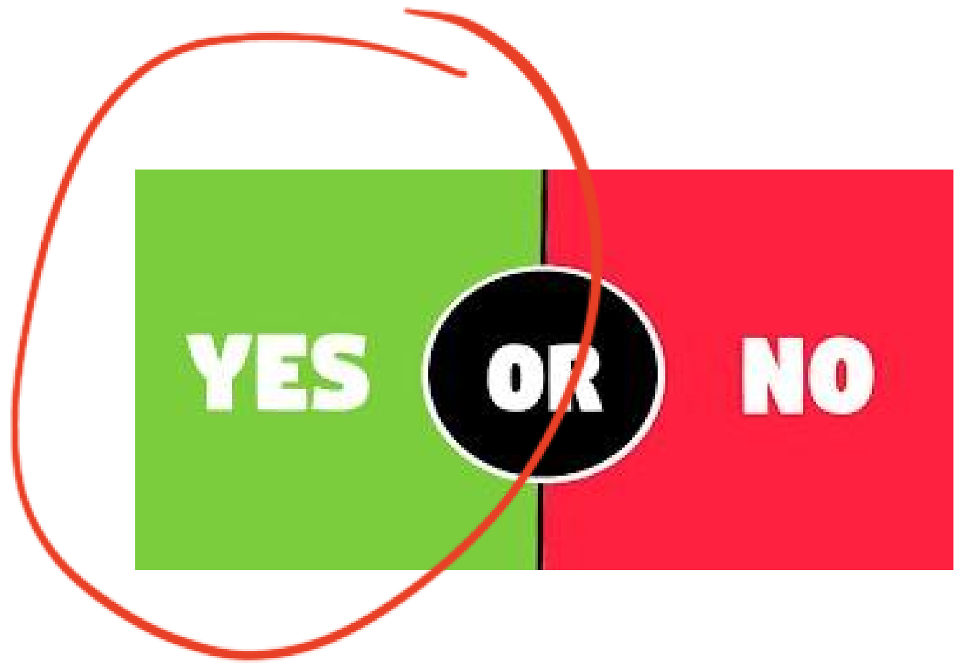
set $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$(1) \iff \langle a, x \rangle = b$

Exercise

Exercise (Linear Equations)

Are the following linear equations?

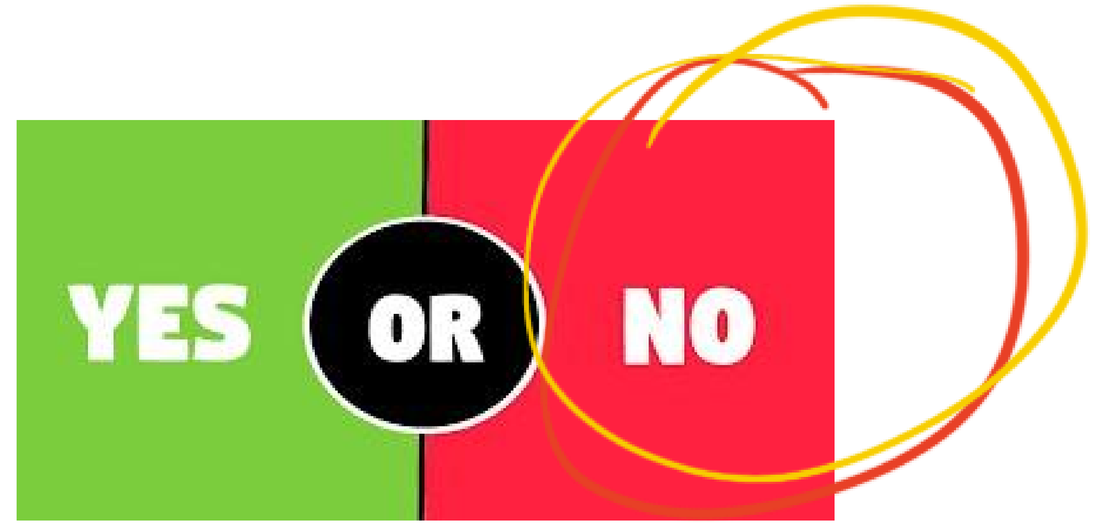


1. $-x_1 + 4x_4 = 2x_2 + 3x_3$

x_1, x_2, x_3, x_4 are variables

Exercise

Exercise (Linear Equations)



Are the following linear equations?

1. $-x_1 + 4x_4 = 2x_2 + 3x_3$ $\quad x_1, x_2, x_3, x_4$ are variables

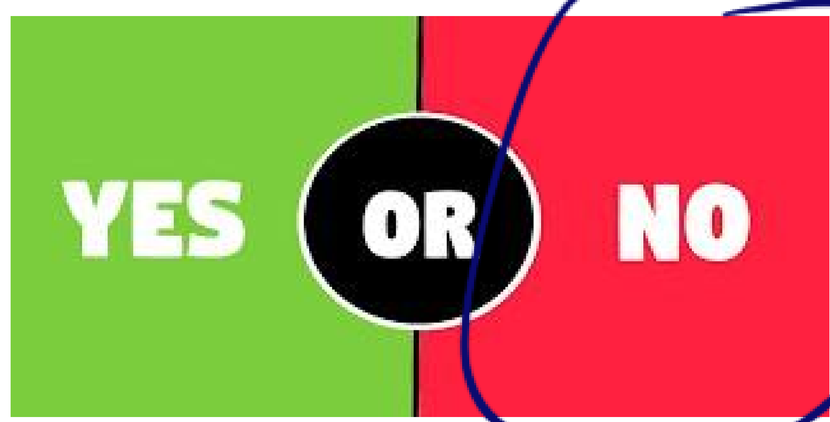
2. $-x_1x_4 = 2x_2 + 3x_3$ $\quad x_1, x_2, x_3, x_4$ are variables

= product between 2 unknowns

Exercise

Exercise (Linear Equations)

Are the following linear equations?



1. $-x_1 + 4x_4 = 2x_2 + 3x_3$ $\quad x_1, x_2, x_3, x_4$ are variables

2. $-x_1x_4 = 2x_2 + 3x_3$ $\quad x_1, x_2, x_3, x_4$ are variables

3. $a_1x_1 + a_2x_2 = b$ $\quad x_1, x_2, a_1, a_2, b$ are variables

it is not a linear equation on the variables a_1, x_1, a_2, x_2, b (but it is a lin. equ. on x_1, x_2)

System of Linear Equations

Definition (System of Linear Equations)

An $m \times n$ **system of linear equations** is a *collection* of m linear equations with n variables

m

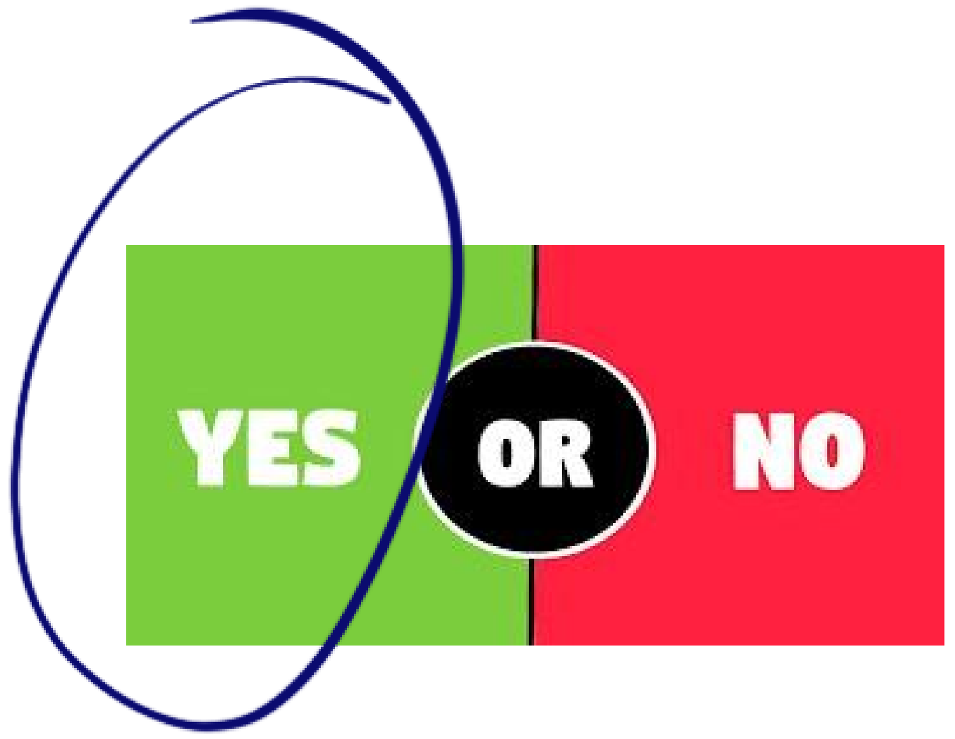
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where all a_{ij} and b_i are real numbers and $x_1, x_2, \ldots, x_n$ are variables

(A system of linear equations can be called a **linear system** for short)

Exercise

Exercise (Linear Equations)



Is the following a linear system?

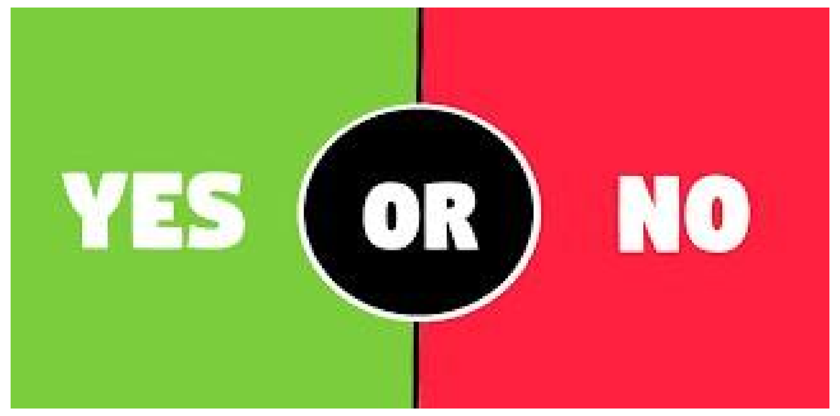
$$-a_{11} + 4a_{12} = 2a_{13} + 3a_{14}$$

$$5a_{11} + 3a_{12} = a_{13} + 3a_{14}$$

Where $a_{11}, a_{12}, a_{13}, a_{14}$ are variables

Exercise

Exercise (Linear Equations)



Is the following a linear system?

$$-a_{11} + 4a_{12} = 2a_{13} + 3a_{14}$$

$$5a_{11} + 3a_{12} = a_{13} + 3a_{14}$$

Where $a_{11}, a_{12}, a_{13}, a_{14}$ are variables

It is critical to know what are the **variables (unknowns)**!

(such as the weight vectors in our movie preference example)

What is New in this Course?

You learned these in middle school (or even primary school).

What more can we study? What will be new?

1. Practice.

   You can solve system of equations in 2 variables.
   What about 5 variables? What about 100 variables?
   What is a general method to solve an any-variable system? (For computers)

2. Theory.

   Does your general method always work? Can you prove it?
   is there always a solution? a unique sol°? → No

First step: To answer these questions, we need to rewrite system of equations with vectors and matrices.

Row-Vector Form of System

(F1) scalar form

$$x_1 + x_2 = 14$$
$$2x_1 + 4x_2 = 36$$

$\xrightarrow{\text{row-reduction}}$

(F2) Row vector form

$$\begin{matrix} (x_1, x_2) \bullet (1, 1) \\ (x_1, x_2) \bullet (2, 4) \end{matrix} = \begin{matrix} 14 \\ 36 \end{matrix}$$

dot product

Can we make the form even simpler?

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 14 \\ 36 \end{bmatrix}$$

Define $Ax \triangleq \begin{pmatrix} (1,1) \cdot x \\ (2,4) \cdot x \end{pmatrix}$ ← vector with 2 entries. , then the system becomes

(F0) matrix form: $Ax = b$

Column-Vector Form

$$x_1 + x_2 = 14$$
$$2x_1 + 4x_2 = 36$$

Another way of writing the equations: Column-vector form

(F3) Column vector form

$$\underbrace{x_1}_{\substack{\text{Scalar} \\ \in \mathbb{R}}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 36 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 4x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_1 + 4x_2 \end{pmatrix}.$$

Four Forms of Linear Systems

Scalar form

$$\begin{cases} x_1 + x_2 = 16, \\ 2x_1 + 4x_2 = 36. \end{cases}$$

Row-vector form

(Unknown vector satisfies n linear equations simultaneously)

Column-vector form

(Unknown combination of columns produces vector b)

Matrix form

(Given matrix times unknown vector produces b)

Four Forms and Matrix-Vector Product

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 14 \\ 36 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 = 16, \\ 2x_1 + 4x_2 = 36. \end{cases} \quad \longleftrightarrow \quad Ax = b$$

Define Ax = $\begin{pmatrix} x_1 + x_2 \\ 2x_1 + 4x_2 \end{pmatrix}$

$$\begin{cases} [1, 1] \cdot [x_1, x_2] = 16, \\ [2, 4] \cdot [x_1, x_2] = 36. \end{cases} \quad \longleftrightarrow \quad Ax = b$$

Define Ax = $\begin{pmatrix} (1,1) \cdot (x_1, x_2) \\ (2,4) \cdot (x_1, x_2) \end{pmatrix}$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2 = \begin{bmatrix} 16 \\ 36 \end{bmatrix}. \quad \longleftrightarrow \quad Ax = b$$

Define Ax = $\begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 4 \end{pmatrix} x_2$.

different ways of writing the same thing $\rightarrow$ flexibility.

Three Definitions of Matrix-Vector Product

Ignore equations for a while. Summarize the last page.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Definition 1: $A\mathbf{x}$ =

Definition 2: $A\mathbf{x}$ =

Definition 3: $A\mathbf{x}$ =

Claim: Three definitions are equivalent.

Next, we extend these definitions to general matrix and vectors..

Part II Matrix-Vector Product & Four Forms of Linear Systems

Textbook v5: Sec. 1.3 (only first half) and Sec. 2.1

Matrix Definition

Definition 4.1 (Matrix)

An $m \times n$ **matrix** A is a rectangular array (= table) of numbers with m *rows* and n *columns* in the following form:

$$A = \begin{array}{cc} & n \\ m & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} =: (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}.$$

where all a_{ij} are scalars.

Def 4.2 (Dimension):

For an $m \times n$ **matrix,** the dimension of A is $m \times n$ (reads "m by n").
For an $m \times 1$ vector, the dimension of the vector is m, or $m \times 1$.

The dimension of $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ is 2×2

Matrix: Example and non-example

= parenthesis.

Often use square bracket

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0.5 & -3.7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix}$$

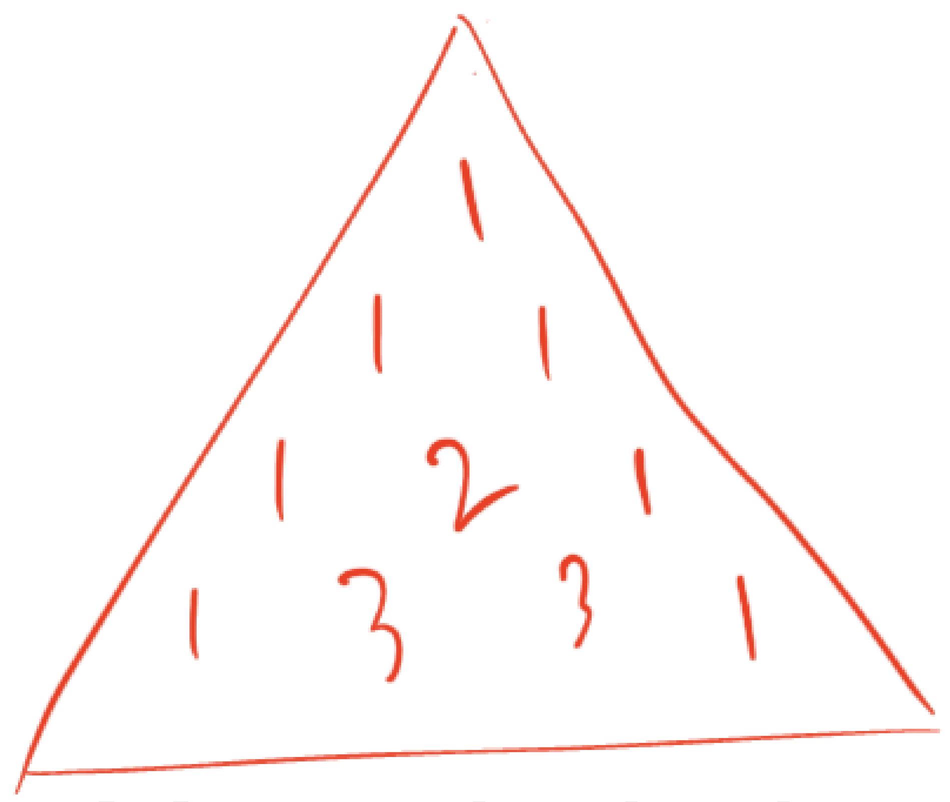
Can also use round bracket

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0.5 & -3.7 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{pmatrix}$$

Non-examples:



Yang-hui triangle

round-table

Matrix Notation

$$A = i \begin{pmatrix} & \vdots^{j} & \\ \cdots & a_{ij} & \\ & & \end{pmatrix}$$

- For a matrix A, a_{ij} is called the (i, j)-th entry (element) of A

Sometimes entry denoted A_{ij} instead of a_{ij}

- Matrices are denoted by A, B, C, ...
- When $m = n$, A is called a *square matrix;* a *rectangular matrix o.w.*
- When all entries are zeros A is called a *zero matrix*

(similar to def of 0 vector)

(similar to zero vector)

$A = 0$:

$\dim A = 3 \times 4$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix v.s. Vector v.s. Scalar

- When $m = 1$, A is a row vector $v = (1, 2 \dots, 3)$
- When $n = 1$, A is a column vector $v = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

Matrix: general $m \times n$

column vector   row vectors

Vector: $\boxed{m \times 1}$ or $\boxed{1 \times n \text{ matrix}}$

Scalar: 1×1 matrix

Remark: In python, scalar and 1 × 1 matrix are different!

e.g. scalar: 3.5 v.s. 1 × 1 matrix: [[3.5]]

（Easily causes bug if you don't know this!）

Matrix Conventions

$(i,j)^{th}$ entry of A.

- Column of a matrix $\boxed{a_j} = \begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix}$

j^{th} column of A.

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots \\ \vdots & \vdots & & \vdots & \\ A_{m1} & A_{m2} & & A_{mj} & \end{pmatrix}$$

"upper index"

- Row of a matrix $a^{(i)} = \begin{bmatrix} A_{i1} & \cdots & A_{in} \end{bmatrix}$

$= i^{th}$ row of A

The skill to identify matrices' columns and matrices' rows is important for the future, but often ignored!

Matrix v.s. Column Vectors: Example of 2 by 3 Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

set of 2x3 matrices

Express A with its columns

Incorrect answer: $\cancel{A = a_1 + a_2} + a_3 \cdot \left(= \begin{pmatrix} 6 \\ 15 \end{pmatrix} \in \mathbb{R}^2 \neq \mathbb{R}^{2\times 3} \right)$

OK but imperfect answer: $A = (a_1 \quad a_2 \quad a_3)$

Correct answer: $A = (a_1 \, , \, a_2 \, , \, a_3)$

Matrix v.s. Column Vectors: General

Write matrix in terms of column vectors.

Suppose the columns of A are: $a_1 = \begin{pmatrix}1\\4\end{pmatrix}$ $a_2 = \begin{pmatrix}2\\5\end{pmatrix}$ $a_3 = \begin{pmatrix}3\\6\end{pmatrix}$

Then A= $\left(a_1, \ a_2, \ a_3 \right)$

Observation:

Matrix can be written as a row vector with each entry

being a column vector.

i.e., Matrix = row vector of column vectors.

(This understanding will be formalized when talking about block matrix)

Matrix v.s. Row Vectors: Example of 2 by 3 Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$a^{(1)} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad a^{(2)} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

Then A= $\begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix}$

i.e., Matrix = column vector of its rows.

Definition of Matrix-Vector Product

Definition 3.1 (Matrix-vector product definition)

Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ n and A be an $m \times n$ matrix $A = (a_{ij})$,

where $i = 1, \ldots, m$; $j = 1, \ldots, n$,

The matrix-vector product Au is an $m \times 1$ column vector:

$$\underset{m\times n \;\; n}{Au} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}u_j \\ \sum_{j=1}^{n} a_{2j}u_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}u_j \end{bmatrix} . \; m! \; = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ \\ a_{m1}u_1 + \cdots + a_{mn}u_n \end{bmatrix}$$

Remark: Dimension checking $\quad A\,u = v$

$$\begin{array}{ccc} A & u & v \\ ? & n \times 1 & m \times 1 \end{array}$$

Example: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}$ $\quad u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad Au = \begin{pmatrix} 1(-1) + 2 \cdot 1 \\ 0(-1) + 1 \cdot 1 \\ 0(-1) + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Special Case: Row * Column Vectors

$$a = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Can we multiply them?

Answer 1: Yes, by Definition of inner product: $\langle a, x \rangle = a_1 x_1 + a_2 x_2$

PB: inner product of row and column vectors are not well defined.

Answer 2: a is a vector, but also a 1x2 matrix.

view it as matrix-vector product.

$$\underset{1\times 2}{a}\ \underset{2\times 1}{x} = \underset{1\times 1}{a_1 x_1 + a_2 x_2}$$

Relation of Inner Product and Matrix-Vector Product

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Inner product: $\langle x, y \rangle =$ $x_1 y_1 + x_2 y_2.$

Matrix-vector product of y and x: Invalid.

$\cancel{xy}$ no meaning

2x1 2x1

Matrix-vector product of x^{T} **and** y**:**

$$x^{\mathsf{T}} y = \quad x_1 y_1 + x_2 y_2.$$

$$(x_1, x_2) \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Two definitions match: $\boxed{\langle x, y \rangle = x^T y}$

This is why we can denote the inner product as $x^{\mathsf{T}} y$

Row-form of Matrix-Vector Product

Property 1 (Row-form of Matrix-vector product)

Let A be an $m \times n$ matrix and $\mathbf{b}$ be an $n \times 1$ column vector. Suppose the matrix A can be represented

$$A = \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \vdots \\ \mathbf{a}^{(m)} \end{bmatrix} \quad 1 \times m$$

Then

$$A\mathbf{b} = \begin{bmatrix} \mathbf{a}^{(1)}\mathbf{b} \\ \mathbf{a}^{(2)}\mathbf{b} \\ \vdots \\ \mathbf{a}^{(m)}\mathbf{b} \end{bmatrix}$$

Recall

$$a^{(j)} = (a_{1j}, \ldots, a_{nj}).$$

$$Ab \overset{def}{=} \begin{pmatrix} \boxed{\sum_{j=1}^{n} a_{1j} b_j} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} b_j \end{pmatrix} = \begin{pmatrix} a^{(1)} \cdot b \\ \vdots \\ a^{(m)} \cdot b \end{pmatrix}$$

$$= a^{(1)} \cdot b$$

Column-form of Matrix-Vector Product

Property 2 (Column-form of Matrix-vector product)

Let A be an $m \times n$ matrix and $\mathbf{w}$ be an $n \times 1$ column vector. Suppose A is represented as:

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n],$$

Then $$A\mathbf{w} = w_1\mathbf{a}_1 + w_2\mathbf{a}_2 + \cdots + w_n\mathbf{a}_n$$

Aw is a linear combination of columns.

Very important observation!

More important than the row-form, in the future parts of the course!

More Examples

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 5 \\ -2 & 1 & 3 & 0 & -1 \\ 0 & 7 & -1 & -2 & 4 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ -1 \end{bmatrix},$$

do it at home.

3x5

5x1

$$\underset{\underline{3\times 5}\ \underline{5\times 1}}{A\,\mathbf{u}} = \begin{bmatrix} 1*1+4*(-2)+2*0+3*5+5*(-1) \\ (-2)*1+1*(-2)+3*0+0*5+(-1)*(-1) \\ 0*1+7*(-2)+(-1)*0+(-2)*5+4*(-1) \end{bmatrix}$$

$$\hookrightarrow A\,\mathbf{u} = \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix} \in \mathbb{R}^3$$

5 columns of A.

Or: $$A\,\mathbf{u} = \underline{1} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \underline{(-2)} \cdot \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + \underline{0} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \underline{5} \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + \underline{(-1)} \cdot \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix}$$

= linear combination of columns of A

coefficients are the entries of u.

Part III Idea of Elimination

Partly from Sec. 2.2

Solve System of Linear Equations

Example (Solving a 2×2 system)

$$x - 2y = 1 \qquad (1)$$
$$3x + 2y = 11 \qquad (2)$$

remove x with $(2) - 3 \times (1)$:

$$0x + 8y = 8 \Rightarrow y = 1$$

(1): $x = 1 + 2 = 3$

More General …

Example (Solving a 3×3 system)

$$\begin{aligned} 2x + 4y - 2z &= 2 \quad (1) \\ 4x + 9y - 3z &= 8 \quad (2) \\ -2x - 3y + 7z &= 10 \quad (3) \end{aligned}$$

3 equa°
3 unknowns.

Key Idea: Elimination.

$(2) - 2 \times (1)$: $0 + 1y + 1z = 4$ (4)
$(3) + (1)$: $y + 5z = 12$ (5)

2 equa° ! less
2 unknowns. difficult.

$(4) - (5)$: $-4z = -8 \Rightarrow z = 2$

go up to retrieve x and y.

Idea of Elimination

How to solve n by n system?

First, eliminate one variable by subtracting one equation from others.

① x, ② y

Second, solve the remaining (n-1) by (n-1) system.

Continue the process until getting 1 variable and 1 equation.

Similar to “prove by induction” (归纳法证明).

Matrix and Linear Systems

Definition (Coefficient Matrix)

Given a linear system,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$Ax = b$

The **coefficient matrix** of the system is an $m \times n$ matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m \times n}
$$

Matrix and Linear Systems

Definition (Coefficient Matrix)

Given a linear system,

unknows.

coefficients

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

?

The **coefficient matrix** of the system is an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n}$$

Augmented Matrix

Definition (Augmented matrix)

Given a linear system,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the corresponding augmented matrix is

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Exercise

Consider
$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 - x_2 &= 4. \end{aligned}$$

What is the coefficient matrix? $\longrightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

What is the augmented matrix? $\longrightarrow \left(\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & -1 & 4 \end{array}\right)$

$$\begin{cases} x_2 + 2x_1 = 0 \\ x_1 + 3x_3 = 2 \\ x_4 + x_2 = -1 \end{cases}$$

Augmented matrix :

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

Summary Today

Today, we have learned:

- Formulation of systems of linear equations
- Matrix-vector product and four forms of linear system
- Idea of elimination