Lecture 04

Solving Linear System I: System

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Recall

In the last lecture …

- Definitions of linear equations and systems of linear equations
- Examples of solving 2×2 system of linear equations
- Definition of Matrix-vector product
- Definition of an augmented matrix representation

Today's Lecture

Today … More on System of **Linear Equations!**

give some rules to be able to deal with any situation

(Today only nxn syst.)

After this lecture, you should be able to

- Tell the definition of lower and upper triangular matrices
- Tell what are elementary row operations, and why they are allowable
- Solve a linear system (square system) using Gaussian Elimination

Part I Gauss-Jordan Elimination and Row Operations

Partly from Sec. 2.2

Length: 40-50 mins.

Recall: Augmented Matrix

Definition (Augmented Matrix)

Given a linear system,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the corresponding augmented matrix is:

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Overdetermined, Underdetermined and Square

Definition (System of Linear Equations)

An $m \times n$ system of linear equations is
(m equations and n unknowns).

(1) **overdetermined system** if $m > n$
(too many conditions → possibility that there are no solutions)

(2) **underdetermined** system if $m < n$
(possible that there is an infinity of sol°)

(3) **square system** if $m = n$

Today we solve an $n \times n$ System (square)

(also possible in this case that there are an infinity or no solutions) ⚠

$x_1 = 1$
$x_1 = 2$
$m = 2,\ n = 1$
→ no sol°

$x_1 + x_2 = 1$
$m = 1,\ n = 2$
→ more than 2 sol°

$\begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases}$
or
$\begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases}$
→ many other sol°

An $m \times n$ matrix is

(1) tall, if $m > n$

(2) wide, if $n > m$

(3) square, if $n = m$

Special Matrices

Definition (Lower Triangular Matrix)

A square matrix of the form

$$L = \begin{bmatrix} \ell_{1,1} & 0 & \cdots & \cdots & 0 \\ \ell_{2,1} & \ell_{2,2} & \ddots & & \vdots \\ \ell_{3,1} & \ell_{3,2} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \ell_{n,1} & \ell_{n,2} & \dots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

ex: 3×3 matrix:

$$L = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

Allowed

is called a **lower triangular matrix.**

Mathematical definition: $L_{ij} = 0$, for any $1 \le i < j \le n$.

i: row index

j: column index.

(try to redo it without the answer)

Special Matrices

Definition (Upper Triangular Matrix)

A square matrix of the form

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

is called an **upper triangular matrix**

Mathematical definition: $U_{ij} = 0$, for any $1 \le j < i \le n$.

Special Matrices

Definition (Diagonal Entry)

For a square matrix A each entry $A_{i,i}$ is called a diagonal entry of

$$L = \begin{pmatrix} l_{11} & & l_{1n} \\ l_{21} & & \\ \vdots & & \\ l_{n1} & & l_{nn} \end{pmatrix}$$

Definition (Diagonal Matrix)

A square matrix D satisfying $D_{ij} = 0, \quad \forall i \neq j$
is called a diagonal **matrix.**

Ex:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

allowed to have 0 on the diagonal.

Gaussian Elimination for 2*2 System: Matrix View

Equation view

systematize the resolution
= Should work in all cases.

⓪
$$\begin{cases} x_1 + x_2 = 12 & (1) \\ 2x_1 + 4x_2 = 38. & (2) \end{cases}$$

① remove x_1 in one equation:
$$\begin{cases} x_1 + x_2 = 12 \\ 0 + 2x_2 = 14 & (2) - 2(1) \end{cases}$$

② ensure that the coefficient in front of x_2 is 1
$$\begin{cases} x_1 + x_2 = 12 \\ 0 + x_2 = 7 & \frac{1}{2} \times (2) \end{cases}$$

③ remove x_2 in first equation
$$\begin{cases} x_1 + 0 = 5 & (1) - (2) \\ 0 + x_2 = 7 \end{cases}$$

solu°: $x_1 = 5$ $x_2 = 7$

more efficient expressions → less mistakes

Later just work with:

Augmented Matrix view

⓪
$$\left(\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 4 & 38 \end{array}\right) \begin{array}{c} (1) \\ (2) \end{array}$$

pivot

①
$$\left(\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 2 & 14 \end{array}\right) \quad (2) \leftarrow (2) - 2(1)$$

triangular matrix

②
$$\left(\begin{array}{cc|c} 1 & 1 & 12 \\ 0 & 1 & 7 \end{array}\right) \quad (2) \leftarrow \frac{1}{2}(2)$$

pivot

Diagonal matrix.

③
$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \end{array}\right) \quad (1) \leftarrow (1) - (2)$$

x_1

x_2

Gaussian Elimination: 3 by 3 System

Step 1: Forward Elimination (Equation)

→ get a triangular matrix

$$\begin{aligned} x + \ y + \ z &= 6 \quad (1) \\ x + 2y + 2z &= 9 \quad (2) \\ x + 2y + 3z &= 10 \quad (3) \end{aligned}$$

$$\begin{aligned} x + y + z &= 6 & & (1) \\ 0 + y + z &= 3 & (2)-(1) \quad & (2) \\ 0 + y + 2z &= 4 & (3)-(1) \quad & (3) \end{aligned}$$

$$\begin{aligned} x + y + z &= 6 \\ y + z &= 3 \\ z &= 1 \quad (3) - (2) \end{aligned}$$

Step 1: Forward Elimination (Matrix)

pivot

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 9 \\ 1 & 2 & 3 & 10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1^{\text{pivot}} & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

upper triangular matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Gaussian Elimination

Step 2: Backward Substitution (Scalar)

$$x + y + z = 6$$
$$y + z = 3$$
$$z = 1$$

1
$$\begin{cases} x + y + z = 6 \\ y + 0 = 2 \\ z = 1 \end{cases}$$

2
$$\begin{cases} x + 0 + 0 = 3 \\ y + 0 = 2 \\ z = 1 \end{cases}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

BREAK → 15:22

Step 2: Backward Substitution (Matrix)

remove

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

pivo

⚠ same operations on 2 sides

1
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

pivots.

2
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

(1). (2). (3)

(here very simple systems).

Review: Gaussian Elimination for “Good” Systems

Pipeline

$$\left(\begin{array}{ccc|c} a_{11} & & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & & a_{mn} & b_m \end{array}\right)$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

⚠ : same operaᵒ on right and left!

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & \times & \cdots & \times & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \times & \cdots & \times & \times \end{array}\right]$$

triangular form

$$\left[\begin{array}{ccccc|c} \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & \times & \times & \times & \vdots \\ & 0 & & \times & \times & \vdots \\ & & & & \times & \times \end{array}\right]$$

↓ diagonal form.

$$\left[\begin{array}{ccccc|c} \times & & & & & \times \\ & \times & & 0 & & \times \\ & & \times & & & \vdots \\ & 0 & & \times & & \vdots \\ & & & & \times & \times \end{array}\right]$$

diagonal matrix with diagonal entries = 1

$$\left[\begin{array}{ccccc|c} 1 & & & & & c_1 \\ & 1 & & 0 & & c_2 \\ & & \ddots & & & \vdots \\ & 0 & & 1 & & \vdots \\ & & & & 1 & c_m \end{array}\right]$$

Solution:

$$\begin{cases} x_1 + 0 \cdots + 0 = c_1 \\ 0 + x_2 + 0 \cdots + 0 = c_2 \\ \vdots \\ 0 + \cdots + 0 + x_n = c_m . \end{cases}$$

Everything works well if you have non zero pivots

(Think: does it always work?) ⟶ answer detailed in next lecture

Summary of GE for Solving Square Systems

Step 1: Forward Elimination. → triangular matrix

Perform elementary row operations and try to get an upper triangular matrix.

Step 2: Backward substitution → diagonal matrix.

Perform elementary row operations and try to get a diagonal matrix.

Assumption 1 At each iteration of the forward elimination, the pivot is nonzero.

no proof given

Claim 1 Under Assumption 1, we can get a diagonal matrix at the end of Step 2.

Corollary 1 Under Assumption 1, the system has a unique solution.

This assumption may not hold for some problems; will discuss later.

Part II Elementary Row Operations

- Three elementary row operations
- Why these operations, not others?

Length: 10-15 mins.

Review

Solving a 2×2 system:

$$\overset{1}{\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 4 & 5 & 14 \end{array}\right]} \xRightarrow[(2) \leftarrow (2) - 4(1)]{} \overset{2}{\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right]} \xRightarrow[(2) \leftarrow -\frac{1}{3}(2)]{} \overset{3}{\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right]} \xRightarrow[(1) \leftarrow (1) - (2)]{} \overset{4}{\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]}$$

"scalar"
= real number

What are the key steps?

just 3 kinds of operations:

(1) Multiply a row by a **non-zero** scalar

(2) Add to one row a scalar multiple of another

(3) Swap the positions of two rows

only performed on rows!

Allowable Operations on Equations

(1) [**Multiplication**] Multiply an equation by a **non-zero** scalar

$$2x_1 + 4x_2 = 38 \quad \xrightarrow{\times \frac{1}{2}} \quad x_1 + 2x_2 = 19$$

(2) [**Addition**] Add to one equation a scalar multiple of another

$$\begin{cases} x_1 + x_2 = 12, \\ 2x_1 + 4x_2 = 38 \end{cases} \quad \xrightarrow{(2) \leftarrow (2) - 2(1)} \quad \begin{cases} x_1 + x_2 = 12 \\ 0 + 2x_2 = 14 \end{cases}$$

(3) [**Interchange**] Swap two equations

$$\begin{cases} x_1 + x_2 = 12, \\ 2x_1 + 4x_2 = 38 \end{cases} \quad \rightarrow \quad \begin{cases} 2x_1 + 4x_2 = 38 \\ x_1 + x_2 = 12 \end{cases}$$

(useful to have non zero pivots)

Operations on linear equations!

Allowable Operations on Rows

Definition (Elementary Row Operations) (初等行变换)

$$A = \begin{bmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{bmatrix} \quad \in \mathbb{R}^{? \times n}$$

(1) [**Multiplication**] Multiply a row by a **non-zero** scalar

$$a^{(i)} \xrightarrow{\times \lambda \in \mathbb{R}} \qquad a^{(i)} \leftarrow \lambda a^{(i)}$$

(2) [**Addition**] Add to one row a scalar multiple of another

$$a^{(i)} \longrightarrow \qquad a^{(i)} \leftarrow a^{(i)} + \lambda a^{(j)}$$

"lambda" (greek letter).

(3) [**Interchange**] Swap the positions of two rows

$$\cancel{\begin{bmatrix} a^{(i)} \\ a^{(j)} \end{bmatrix}} \longrightarrow$$

$$\begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(i)} \\ \vdots \\ a^{(j)} \\ \vdots \\ a^{(n)} \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i\text{th} \\ \\ \leftarrow j\text{th} \\ \\ \\ \end{array} \qquad \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(j)} \\ \vdots \\ a^{(i)} \\ \vdots \\ a^{(n)} \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i\text{th} \\ \\ \leftarrow j\text{th}. \\ \\ \\ \end{array}$$

Typical Steps

reached this point.

After cleaning the two first columns we have:

pivot

$$\begin{array}{c} \\ \end{array}\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}\begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

$c \hookrightarrow 0$

We want:

$$\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$$

1st idea: $(4) \leftarrow (3) - \frac{c}{a}(4)$: $\begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & a & b \\ 0 & 0 & 0 & d - \frac{c}{a}b \end{pmatrix}$

→ only possible if $a \neq 0$ (pivot $\neq 0$)

$c - \frac{c}{a}a = 0$

$\boxed{\text{What if } a = 0\text{?}}$

- if $c = 0$ then we are happy (already triangular matrix)
- if $c \neq 0$: interchange : $(3) \leftrightarrow (4)$

$$\begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & c & d \\ 0 & 0 & 0 & b \end{pmatrix}$$

Why no operations on columns?

why not: $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 3 & 1 \end{array}\right)$?

clear that there is no meaning when look at equations:

that would mean:

wrong!

$$\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 4x_2 = 1 \end{cases} \quad \cancel{\Leftrightarrow} \quad \begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 3x_2 = 1 \end{cases}$$

Elementary Row Operations Preserves Solution

Exercise (The operations preserve solutions)

Performing elementary operations will create a new system.

Pr~~o~~ve: The new system and the original system has the same solution(s).

We want to preserve equivalence between systems and that is what those operations do

Other Operations

Exercise (Other Operations)

Can the following operations be performed?

(4) Multiply a row by zero

(5) Multiply the coefficients of two equations

why not :
$$\begin{cases} 2x_1 + 3x_2 = 2 \\ x_1 + 2x_2 = 0 \end{cases} \quad \begin{matrix} \longrightarrow \\ \Longrightarrow \\ \cancel{\Longleftarrow} \end{matrix} \quad \begin{cases} 0x_1 + 0x_2 = 0 \; ? \\ x_1 + 2x_2 = 0 \end{cases}$$

because then it is not an equivalence

why not
$$\begin{cases} 2x_1 + 3x_2 = 2 \\ x_1 + 2x_2 = 0 \end{cases} \longrightarrow \begin{cases} 4x_1 + 6x_2 = 4 \\ 2x_1 + 4x_2 = 0 \end{cases}$$

"Gaussian elimina°"

we could do that actually but not useful for GE

Concluding Section

Summary Today

One sentence summary:

Detailed summary:

Questions:

Can we use matrix operations to represent GE?

Yes → next lecture